

ON CLASSICAL GLOBAL SOLUTIONS OF NONLINEAR WAVE EQUATIONS WITH LARGE DATA

SHUANG MIAO, LONG PEI, AND PIN YU

ABSTRACT. This paper studies the Cauchy problem for systems of semi-linear wave equations on \mathbb{R}^{3+1} with nonlinear terms satisfying the null conditions. We construct future global-in-time classical solutions with arbitrarily large initial energy. The choice of the large Cauchy initial data is inspired by Christodoulou's characteristic initial data in his work [2] on formation of black-holes. The main innovation of the current work is that we discovered a relaxed energy ansatz which allows us to prove decay-in-time-estimate. Therefore, the new estimates can also be applied in studying the Cauchy problem for Einstein equations.

CONTENTS

1. Introduction	1
1.1. Historical results	2
1.2. The short pulse data and main results	3
1.3. Notations	5
1.4. Comments on the proof	8
2. Short pulse region	12
2.1. Main a priori estimates	12
2.2. Higher Order Estimates	27
2.3. Existence based on <i>a priori</i> estimates	31
2.4. Improved Estimates on C_δ	31
3. Small data region	34
3.1. Klainerman-Sobolev inequality revisited	34
3.2. A priori energy estimates	36
Acknowledgment	40
References	40

1. INTRODUCTION

We consider the Cauchy problem of the following system of wave equations on \mathbb{R}^{3+1} :

$$\square\phi = Q(\nabla\phi, \nabla\phi). \quad (1.1)$$

Here, $\square = -\partial_t^2 + \Delta$ is the standard wave operator. The function ϕ is vector valued. In fact, ϕ stands for N unknown functions ϕ^I , $I = 1, \dots, N$. The symbol $\nabla\phi$ denotes all possible $\partial_\gamma\phi^I$'s for $\gamma = 0, 1, 2, 3$ and $I = 1, 2, \dots, N$. The nonlinearity $Q(\nabla\phi, \nabla\phi)$ is a quadratic form in $\nabla\phi$ satisfying the *null condition*,

Key words and phrases. large data and null condition and wave equations.

Department of Mathematics, The University of Michigan, Ann Arbor, MI 48109 U.S.A. shmiao@umich.edu.

Department of Mathematical Sciences, Norwegian University of Science and Technology. long.pei@math.ntnu.no.

Yau Mathematical Sciences Center, Tsinghua University, Beijing, China. pin@math.tsinghua.edu.cn.

which will be specified later. The problem of constructing global-in-time solutions for small initial data has been studied intensively in the literature. The purpose of the current paper is to propose a large Cauchy data regime for (1.1) which also leads to global classical solutions.

1.1. Historical results. We discuss briefly the small data theory for (1.1) on \mathbb{R}^{n+1} . The idea is to use the decay mechanism of linear waves, i.e. solutions of $\square\phi = 0$, and treat the nonlinear problem as a perturbation of the linear waves. In dimensions greater than 3, i.e. $n \geq 4$, the pointwise decay rate of linear waves is at least $t^{-3/2}$, which is integrable on for $t \geq 1$. This fast decay rate can be used to prove the small-data-global-existence results; see the pioneering works of Klainerman [5] and [6]. However, in \mathbb{R}^{3+1} , the pointwise decay of the linear wave is merely at the rate t^{-1} which is not integrable. This weak decay rate is not enough to control the nonlinear interaction: F. John [4] showed that there were quadratic forms (which do not satisfy the null condition) such that for arbitrarily small non-zero smooth data, solutions to (1.1) blow up in finite time.

The importance of the null condition was first observed in the breakthrough work [7] by Klainerman, where he proved that small data lead to global-in-time classical solutions if the nonlinearity Q is a *null form*, which will be defined explicitly later, or equivalently, satisfies the null condition. In [1] Christodoulou obtained a similar result based on the conformal compactification of the Minkowski spacetime. Although the approaches in [7] and [1] are very different, both proofs rely on the cancellation structure of null condition, which is absent for general quadratic nonlinearities.

The idea of exploiting the cancellation structure of the null conditions can also be used to handle certain large data problems. In a recent breakthrough in general relativity, Christodoulou [2] rigorously proved for the first time that black-holes can form dynamically from arbitrarily dispersed initial data. The key to this work was the new idea of the "short pulse method". Roughly speaking, this is a choice of special large initial data, called *short pulse* data, so that these large profiles can be propagated along the flow of Einstein vacuum equations. One of the key observations in the proof is still tightly related to the cancellation of the null structure: the profile is only large in certain components and these large components are always coupled with some small components so that their contributions are still manageable. Christodoulou's work has been generalized in [8] by Klainerman and Rodnianski. A key ingredient in their work is the relaxed propagation estimates which allows profiles with more large components.

The ideas used in [2] and [8] have been adapted to the main equation (1.1) by Wang and Yu to construct future-in-time global solutions with large initial data; see [10] and [11]. Their approach is indirect. The authors essentially impose the characteristic data on the past null infinity and solve the inverse scattering problem all the way up to a finite time to construct the initial Cauchy data. Very recently, Yang [12] has obtained a global existence theorem for semi-linear wave equations with large Cauchy initial energy. The largeness in [12] is from a slower decay of the initial data at spatial infinity, but not from the short pulse method.

The aim of the current work is to study the global-in-time behavior of smooth solutions to (1.1) with short pulse data. We give short pulse Cauchy data directly (one should compare with the indirect approach of [10]) and prove that the data lead to future-global-in-time classical solutions for (1.1). We remark that compared to the characteristic data approach in [10], one of the main difficulties is to prove quantitative decay of the solution. This difficulty does not appear in [10], because the data there are radiation fields given on the past null infinity, so that the decay rate is already explicitly given. We will give a more-detailed comparison of the present work and [10] after some necessary notations are introduced.

1.2. The short pulse data and main results. We use (x_0, x_1, x_2, x_3) to denote the standard Cartesian coordinates (t, x, y, z) on \mathbb{R}^{3+1} . In particular, ∂_0 stands for ∂_t . Let $\phi : \mathbb{R}^{3+1} \rightarrow \mathbb{R}^N$ be a vector valued function, and we use ϕ^I to denote its components. We study the Cauchy problem for the following system of nonlinear wave equations

$$\begin{aligned} \square \phi^I &= Q^I(\nabla \phi, \nabla \phi), \quad \text{for } I = 1, 2, \dots, N, \\ (\phi, \partial_t \phi)|_{t=1} &= (\phi_0, \phi_1). \end{aligned} \quad (1.2)$$

The nonlinearities Q^I are *null forms*, i.e. we can write $Q^I(\nabla \phi, \nabla \phi)$ as

$$Q^I(\nabla \phi, \nabla \phi) = \sum_{\substack{0 \leq \alpha, \beta \leq 3, \\ 1 \leq J, K \leq N}} A_{JK}^{\alpha\beta, I} \partial_\alpha \phi^J \partial_\beta \phi^K,$$

and for all *null* vector $\xi \in \mathbb{R}^{3+1}$, i.e. $\xi = (\xi_0, \xi_1, \xi_2, \xi_3)$ satisfying $-\xi_0^2 + \sum_{i=1}^3 \xi_i^2 = 0$, the coefficient matrices $A_{JK}^{\alpha\beta, I}$ satisfy

$$\sum_{\alpha, \beta=0}^3 A_{JK}^{\alpha\beta, I} \xi_\alpha \xi_\beta = 0.$$

For the sake of simplicity, instead of writing all the components, we shall always use ϕ to denote the ϕ^I 's and use $Q(\nabla \phi, \nabla \phi)$ to denote the nonlinearity. In particular, we always write the main equation (1.2) as (1.1). We remark that in order to simplify some of the expressions appearing in the proof of the main theorem, we give the initial data at $t = 1$ rather than $t = 0$. Because of the invariance of the equation under time translations, this is the same as giving data on $t = 0$.

Before describing the short pulse data, we introduce some notations: r and θ are used to denote the usual radial and angular coordinates on \mathbb{R}^3 . Let δ be a small positive constant which will be determined later. We identify the $t = 1$ hypersurface with \mathbb{R}^3 and divide it into three parts:

$$\{t = 1\} = B_{1-2\delta} \cup (B_1 - B_{1-2\delta}) \cup (\mathbb{R}^3 - B_1),$$

where B_r is the ball centered at the origin with radius r .

In the following, $f \lesssim g$ always means there exists a constant C such that $f \leq Cg$ holds. We consider the initial data (ϕ_0, ϕ_1) on $\{t = 1\}$ of (1.2) satisfying the following conditions:

- On $B_{1-2\delta}$, we set $(\phi_0, \phi_1) \equiv (0, 0)$.
- On $B_1 - B_{1-2\delta}$,

$$\|\nabla^k (\phi_1 + \partial_r \phi_0)\|_{L^\infty} \lesssim \delta^{1/2-k}, \quad (1.3)$$

and

$$\|\nabla^k \phi_0\|_{L^\infty} + \|\nabla^{k-1} \phi_1\|_{L^\infty} \lesssim \delta^{1/2-k} \quad (1.4)$$

for any positive integer $1 \leq k \leq 20$.

- On $\mathbb{R}^3 - B_1$, $(\phi_0, \phi_1) \equiv (0, 0)$.

In particular, the following data satisfies (1.3) and (1.4):

$$\phi_0(r, \theta) = \delta^{1/2} \psi_0 \left(\frac{1-r}{2\delta}, \theta \right), \quad \phi_1(r, \theta) = \delta^{-1/2} \psi_1 \left(\frac{1-r}{2\delta}, \theta \right). \quad (1.5)$$

Here $\psi_0(s, \theta)$ and $\psi_1(s, \theta)$ are smooth functions supported in $(0, 1)$ with respect to their first argument s . Moreover, ψ_0 and ψ_1 satisfy

$$\|(\partial_t + \partial_r) \phi\|_{L^\infty(\Sigma_1)} \lesssim \delta^{50}, \quad (1.6)$$

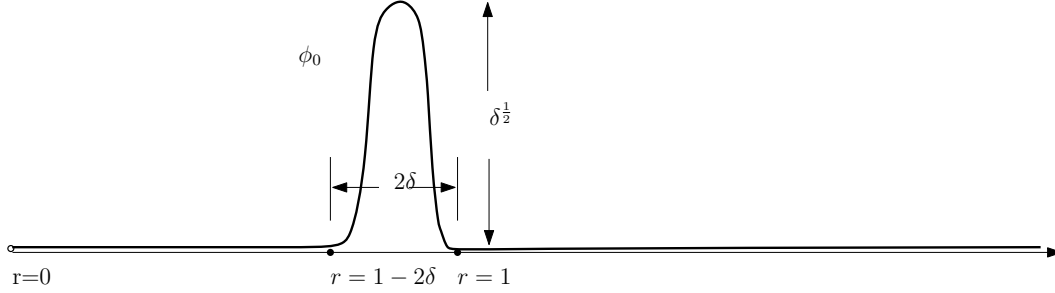
and

$$\|\partial_t^2 \phi - \partial_r^2 \phi\|_{L^\infty(\Sigma_1)} \lesssim \delta^{50}. \quad (1.7)$$

(1.6) and (1.7) can be achieved because we have the freedom to choose ψ_0, ψ_1 and there are two constraints to satisfy. Using an induction argument and in view of (1.6), (1.7), it is straightforward to see that for $k \leq 40$, we have

$$\|(\partial_t + \partial_r)^k \phi\|_{L^\infty(\Sigma_1)} \lesssim \delta^{1/2}. \quad (1.8)$$

For a fixed value of θ , the graph of ϕ_0 versus the variable r is as follows:



The pulse-like shape of the graph explains the name “short pulse” used for this data. The width of the pulse is 2δ and its amplitude is $\delta^{1/2}$, which is very large relative to the width if δ is small.

The choice of $\phi_1(r, \theta)$ looks obscure and artificial in the above form. In fact, we have a natural geometric explanation of this choice, which can also serve as heuristics to understand why one expects a global-in-time solution.

Remark 1.1 (Geometric / Physical interpretation). *In terms of the solution ϕ , it is easy to observe from our initial data (1.3) that*

$$|(\partial_t + \partial_r)\phi|_{t=1} \lesssim \delta^{1/2}, \quad |\nabla \phi|_{t=1} \lesssim \delta^{1/2}. \quad (1.9)$$

We will prove that if δ is small enough, the smallness indicated by (1.9) can be propagated by showing

$$\|(\partial_t + \partial_r)\phi\|_{L^\infty(\Sigma_t)} \lesssim \delta^{1/2} t^{-2}, \quad \|\nabla \phi\|_{L^\infty(\Sigma_t)} \lesssim \delta^{1/4} t^{-2}. \quad (1.10)$$

Here we use Σ_t to denote the hypersurface $\{t = t\}$. Recall that $L = \partial_t + \partial_r$ is the normal (with respect to the Minkowski metric!) of the outgoing light cones $t - r = \text{constant}$ in \mathbb{R}^{3+1} . If we integrate $|(\partial_t + \partial_r)\phi|^2 + |\nabla \phi|^2$ on such an outgoing light cone C , the quantity

$$\int_C |(\partial_t + \partial_r)\phi|^2 + |\nabla \phi|^2 d\mu_C$$

measures the incoming energy through this light cone. Therefore, since δ will be eventually very small, the choice of ϕ_1 is to keep the incoming energy as small as possible. Intuitively, we expect all the energy will be emanated in the outgoing direction so that the solution ϕ disperses.

We now explain in what sense the short pulse data are large. It appears that the short pulse data is at least small in the L^∞ sense due to the presence of the factor δ . First of all, we notice that the L^∞ norm is irrelevant since we may always add a constant to get a new solution for (1.2). The size of the data should be measured at least on the level of first derivatives. Secondly, we notice that, if we take derivatives in the ∂_r direction many times, the data can be extremely large in the L^∞ sense, because each ∂_r derivative will bring out a δ^{-1} factor from the first argument of ϕ_0 or ϕ_1 .

A more natural way to see the largeness of the data is to consider the energy spaces, i.e. the Sobolev spaces $H^k(\mathbb{R}^3)$. The critical H^s -exponent (with respect to scaling) of (1.2) is $\frac{3}{2}$. Therefore, the 0th order energy $\mathcal{E}_0 = \int_{\mathbb{R}^3} |\nabla_x \phi_0|^2 + |\phi_1|^2 dx$ is subcritical and the 1st order energy $\mathcal{E}_1 = \sum_{i=1}^3 \int_{\mathbb{R}^3} |\nabla_x \partial_i \phi_0|^2 + |\partial_i \phi_1|^2 dx$ is supercritical. Here we use ∇_x to denote spatial gradient.

Remark 1.2 (Largeness of short pulse data). *We can compute the 0th order energy \mathcal{E}_0 and the 1st order energy \mathcal{E}_1 as follows:*

$$\begin{aligned}\mathcal{E}_0 &\sim \|\nabla_x \phi_0\|_{L^2}^2 + \|\phi_1\|_{L^2}^2 \sim 1, \\ \mathcal{E}_1 &\sim \|\nabla_x^2 \phi_0\|_{L^2}^2 + \|\nabla_x \phi_1\|_{L^2}^2 \sim \delta^{-2}.\end{aligned}$$

Since \mathcal{E}_0 and \mathcal{E}_1 are subcritical and supercritical, respectively, we can not make both of them small by the scaling invariance of the equation. It is in this sense that the data are large at the level of energy.

Moreover, for all $k \geq 0$, we can show that

$$\mathcal{E}_k = \int_{\mathbb{R}^3} |\nabla_x^{k+1} \phi_0|^2 + |\nabla_x^k \phi_1|^2 dx \sim \delta^{-2k}.$$

We note in passing that the higher order energies can be extremely large. Also, we remark that the symbol \sim depends only on an absolute constant.

We are now ready to state the main theorem of the paper:

Main Theorem. *For any given pair of short pulse data (ϕ_0, ϕ_1) as above, let us consider the following system of wave equations*

$$\begin{aligned}\square \phi^I &= Q^I(\nabla \phi, \nabla \phi), \quad \text{for } I = 1, 2, \dots, N, \\ (\phi, \partial_t \phi)|_{t=1} &= (\phi_0, \phi_1).\end{aligned}$$

where the Q^I 's are null forms.

Then there exists an absolute positive number δ_0 , so that for all $\delta < \delta_0$, the above Cauchy problem admits a unique smooth solution ϕ with lifespan $[1, +\infty)$. Moreover, when $t \rightarrow \infty$, the nonlinear wave ϕ scatters.

1.3. Notations. We review the basic geometry of Minkowski space \mathbb{R}^{3+1} . In particular, we discuss the standard double null (cone) foliations on \mathbb{R}^{3+1} which will play a central role for the energy estimates.

Let $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$. We define two optical functions u and \underline{u} as follows

$$u = \frac{1}{2}(t - r), \quad \underline{u} = \frac{1}{2}(t + r).$$

For a given constant c , we use C_c to denote the level surface $u = c$ with an extra constraint that $t \geq 1$ (since we will construct a future-global-in-time solution starting from the initial hypersurface $\{t = 1\}$). According to the different value of u , we use also C_u to denote these hypersurfaces. These are called outgoing light cones. Thus, $\{C_u | u \in \mathbb{R}\}$ defines a foliation of $\mathbb{R}_{t \geq 1}^{3+1}$. We also call this foliation null because each leaf C_u is a null hypersurface with respect to the Minkowski metric.

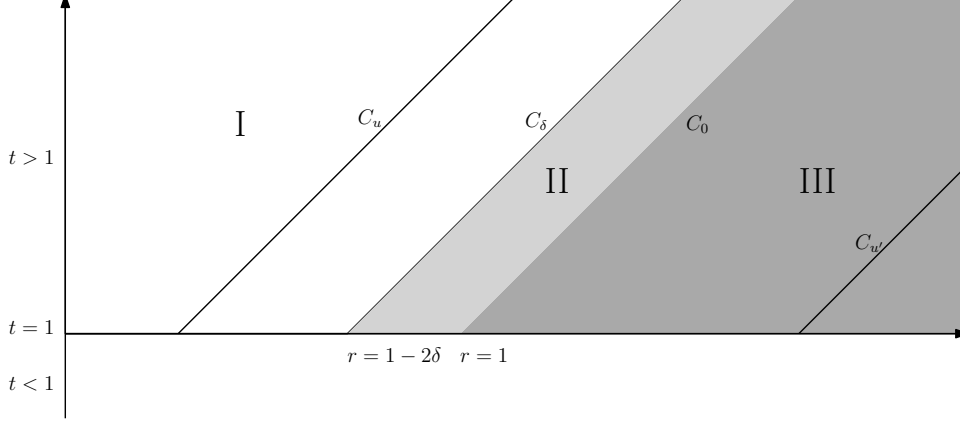
Similarly, using the level sets of the optical function \underline{u} , we define another null foliation of $\mathbb{R}_{t \geq 1}^{3+1}$, denoted by $\{\underline{C}_{\underline{u}} | \underline{u} \in \mathbb{R}\}$. Each $\underline{C}_{\underline{u}}$ is a truncated incoming light cone. The intersection $C_u \cap \underline{C}_{\underline{u}}$ is a round 2-sphere with radius $\underline{u} - u$, denoted by $S_{\underline{u}, u}$. We say that the two foliations $\{\underline{C}_{\underline{u}} | \underline{u} \in \mathbb{R}\}$ and $\{C_u | u \in \mathbb{R}\}$ form a double null foliation of $\mathbb{R}_{t \geq 1}^{3+1}$.

We recall that the following two null vector fields,

$$L = \partial_t + \partial_r, \quad \text{and} \quad \underline{L} = \partial_t - \partial_r$$

are the normals of (also parallel to) C_u and $\underline{C}_{\underline{u}}$ respectively. In the following, null pair always refers to the pair of two null vector fields (L, \underline{L}) .

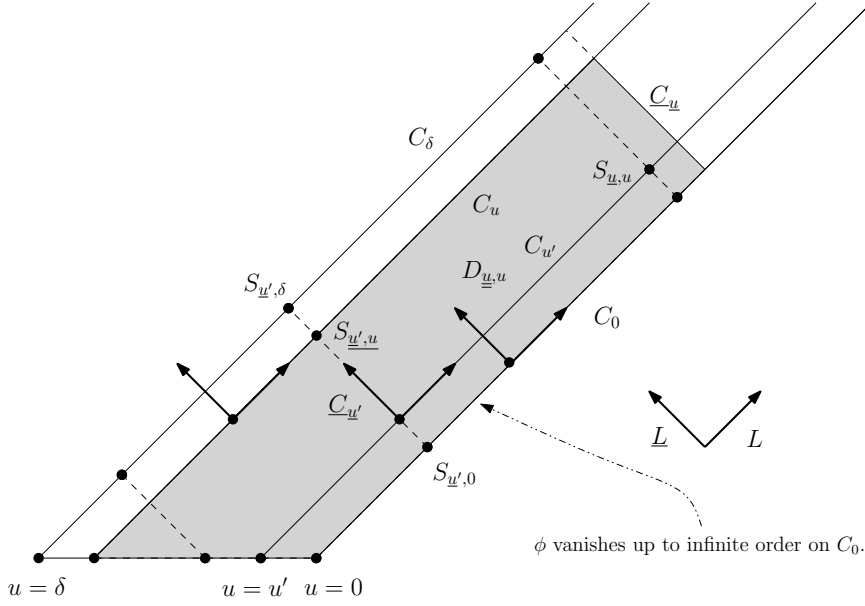
The following picture depicts the outgoing null foliation C_u of $\mathbb{R}_{t \geq 1}^{3+1}$:



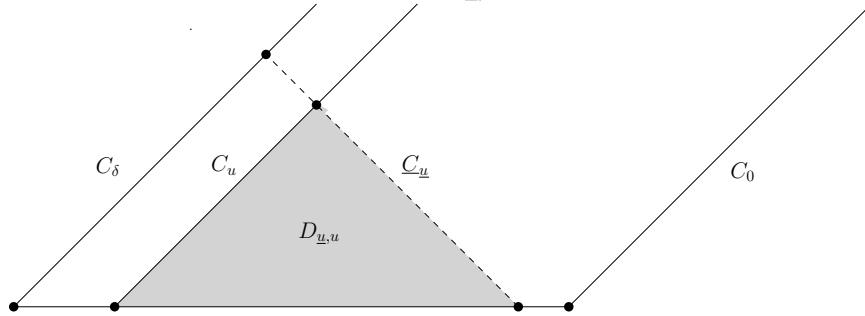
Since the foliation is spherically symmetric, we only draw the t and r components in the schematic diagram. The other pictures in the paper should also be understood in this way. In the above picture, a 45° line denotes an outgoing cone C_u . Two outgoing cones C_0 and C_δ divide \mathbb{R}^{3+1} into three regions: the small data region, i.e. region I in the picture, the short pulse region, i.e. the region with light grey color, region II in the picture, and the region III in the picture, i.e. the region with dark grey color.

Remark 1.3 (Vanishing Property on C_0). *Recall that the short pulse data prescribed on $\{t = 1\}$ in the last subsection are identically zero for $r \geq 1$, therefore, according to the weak Huygen's principle, the solution of the main equation (1.2) vanishes identically in the region III (dark grey). In particular, the solution ϕ (if it exists) and its derivatives vanish on C_0 .*

We now pay more attention to the short pulse region (region II with light grey color). We use $D_{\underline{u}, u}$ to denote the interior of the spacetime region enclosed by the hypersurfaces $\{t = 1\}$, C_0 , C_u and $\underline{C}_{\underline{u}}$, where $u \in [0, \delta]$ and $\underline{u} \geq 1 - u$. The following picture is a schematic diagram for all the notations introduced in this section for the short pulse region.



A dashed 45° segment denote an incoming cone \underline{C}_u . A thickened black point denotes a 2-sphere $S_{\underline{u},u}$. An orthogonal pair of arrow denotes the null vector pair (L, \underline{L}) . A typical picture (if $\underline{u} \geq 1$) of $D_{\underline{u},u}$ is the grey region. If $\underline{u} < 1$, the picture of $D_{\underline{u},u}$ looks like a triangle:



We remark that, for both cases, both $\{C_{u'} | 0 \leq u' \leq u\}$ and $\{\underline{C}_{\underline{u}'} | 1 - u \leq \underline{u}' \leq \underline{u}\}$ foliate $D_{\underline{u},u}$.

In view of Remark 1.1, we also remark that the choice of the short pulse data is also adapted to the double null foliation in the short pulse region: the data is chosen in a way that very little energy propagates in the incoming direction through C_u 's. We expect most of the energy will radiate through the \underline{C}_u 's to the future null infinity.

For a given 2-sphere $S_{\underline{u},u}$, we use g to denote the induced metric from the Minkowski metric on $S_{\underline{u},u}$. The intrinsic covariant derivative on $S_{\underline{u},u}$ is denoted by ∇ . This covariant derivative is closely related to the rotational symmetry of \mathbb{R}^{3+1} . Recall that the infinitesimal rotations are represented by the following three vector fields:

$$\Omega_{ij} = x_i \partial_j - x_j \partial_i, \quad \text{for } 1 \leq i < j \leq 3.$$

We use Ω as a short hand notation for an arbitrary choice from the above vector fields. We also use Ω^2 to denote an operator of the form $\Omega_{i'j'} \Omega_{ij}$; similarly for Ω^n . For a given function ϕ , we use $|\Omega\phi|$ to denote $\sum |\Omega_{ij}\phi|$ and use $|\Omega^2\phi|$ to denote $\sum |\Omega_{ij}\Omega_{i'j'}\phi|$, and so on. Therefore, by a direct computation, we obtain

$$|\Omega\phi| \sim r |\nabla\phi|,$$

where the size $\nabla\phi$ is measured with respect to \mathcal{J} . Moreover, for all n , we have

$$|\Omega^n \phi| \sim_n r^n |\nabla^n \phi|.$$

In the rest of the paper, the number of derivatives that we impose on the solution is a fixed number which does not exceed, say, 30. Therefore the dependence on n in the above inequality is universal.

Remark 1.4. *In the short pulse region II, if δ is sufficiently small, then $|\underline{u}| \sim r$. Therefore, for all n , we have*

$$|\Omega^n \phi| \sim |\underline{u}|^n |\nabla^n \phi|.$$

In particular, for all $p \geq 1$ and n , we have

$$\|\Omega^n \phi\|_{L^p(S_{\underline{u}, u})} \sim |\underline{u}|^n \|\nabla^n \phi\|_{L^p(S_{\underline{u}, u})}.$$

1.4. Comments on the proof. We construct a solution ϕ in three steps:

- **Step 1** We construct ϕ in the short pulse region II.

The initial data for this region are given on initial hypersurface Σ_1 . We expect to see the largeness of the data in the proof. In particular, the \underline{L} derivative of the solution causes a loss of δ^{-1} . This makes the proof difficult and also different from the classical small data problem. The decay of derivatives of ϕ is another difficulty which will be explained in detail in Section 1.4.3.

- **Step 2** Smallness of ϕ on C_δ .

Although ϕ constructed in **Step 1** has large derivatives, we show that the derivatives of ϕ are indeed small on the inner boundary C_δ . This is a key intermediate step: since in next step, the ϕ restricted on C_δ gives initial characteristic data, this step allows one to reduce the problem to a small data problem in region I.

- **Step 3** We construct ϕ in the small data region I.

In region I, the problem is reduced to a small data problem. We can then use the classical approach to construct ϕ .

1.4.1. Vector field method. We will derive energy estimates for the main equation (1.2). Our approach is based on the classical vector field method and we briefly recall the main structure of the method as follows.

Let ϕ be a (scalar) solution for a non-homogenous wave equation $\square\phi = \Phi$ on \mathbb{R}^{3+1} . The energy-stress tensor associated to ϕ is $\mathbb{T}_{\alpha\beta}[\phi] = \nabla_\alpha\phi\nabla_\beta\phi - \frac{1}{2}g_{\alpha\beta}\nabla^\mu\phi\nabla_\mu\phi$ where $g_{\alpha\beta}$ is the Minkowski metric. In particular, in terms of the null pair (L, \underline{L}) , we have $\mathbb{T}[\phi](L, L) = (L\phi)^2$, $\mathbb{T}[\phi](\underline{L}, \underline{L}) = (\underline{L}\phi)^2$ and $\mathbb{T}[\phi](L, \underline{L}) = |\nabla\phi|^2$. Given a vector field X , we use ${}^{(X)}\pi_{\mu\nu} = \frac{1}{2}\mathcal{L}_X g_{\mu\nu}$ to denote its deformation tensor. The energy currents associated to ϕ are defined by $J_\alpha^X[\phi] = \mathbb{T}_{\alpha\mu}[\phi]X^\mu$ and $K^X[\phi] = \mathbb{T}^{\mu\nu}[\phi]{}^{(X)}\pi_{\mu\nu}$. The following divergence identity is the key to the energy estimates:

$$\nabla^\alpha J_\alpha^X[\phi] = K^X[\phi] + \Phi \cdot X\phi. \quad (1.11)$$

In applications, we integrate this identity on the spacetime region. This is equivalent to multiplying $\square\phi = \Phi$ by $X\phi$ and then integrating by parts. This is the reason that we call X a *multiplier* vector field.

In the short pulse region II, we integrate (1.11) on $D_{\underline{u}, u}$. Since ϕ and its derivatives vanish on C_0 , this yields

$$\int_{C_u} \mathbb{T}[\phi](X, L) + \int_{\underline{C}_{\underline{u}}} \mathbb{T}[\phi](X, \underline{L}) = \frac{1}{2} \int_{\Sigma_1} \mathbb{T}[\phi](X, L + \underline{L}) + \iint_{D_{\underline{u}, u}} K^X[\phi] + \Phi \cdot X\phi. \quad (1.12)$$

where Σ_1 is the initial Cauchy hypersurface $\{t = 1\}$.

In the short pulse region II, we will use two multiplier vector fields: $X = \underline{L}$ and $X = \underline{u}^\alpha L$, where the power $\alpha = 1 - \varepsilon_0$ and $\varepsilon_0 \in (0, \frac{1}{2})$ is a given constant. The first plays a similar role to the time vector field ∂_t ; the second plays a similar role to the vector field $S = t\partial_t + r\partial_r$ as a multiplier vectorfield.

For $X = \underline{L}$ and $X = \underline{u}^\alpha L$, the corresponding deformation tensors and energy currents are

- For $X = \underline{L}$, $\pi_{AB} = -\frac{1}{r}\mathcal{J}_{AB}$ and $K = -\frac{1}{r}L\phi \cdot \underline{L}\phi$.
- For $X = \underline{u}^\alpha L$, $\pi_{L\underline{L}} = -\alpha \underline{u}^{\alpha-1}$, $\pi_{AB} = \frac{1}{r}\underline{u}^\alpha \mathcal{J}_{AB}$ and $K = -\frac{\alpha}{2}\underline{u}^{\alpha-1}|\nabla\phi|^2 + \frac{1}{r}\underline{u}^\alpha L\phi \cdot \underline{L}\phi$.

respectively. We remark that indices A and B are used to denote a frame on $S_{\underline{u},u}$ and we only listed the nonzero components of the deformation tensors.

We will also need estimates for higher order derivatives for ϕ . To achieve this, we will commute the main equation (1.2) with certain vector fields, i.e. the *commutator vector fields*. These vector fields are essentially the Lie algebras of the conformal isometries of \mathbb{R}^{3+1} . We list all of them as follows:

$$\mathcal{Z} = \{\Omega_{ij}, \Omega_{0i}, \partial_t, \partial_i, S | i, j = 1, 2, 3, i \neq j\},$$

where $\Omega_{0i} = x_i\partial_t + t\partial_i$ and $S = t\partial_t + r\partial_r = \underline{u}L + \underline{u}\underline{L}$. We also define the *good* and *bad* commutator vector fields:

$$\mathcal{Z} = \mathcal{Z}_g \sqcup \mathcal{Z}_b, \quad \mathcal{Z}_b = \{\partial_t, \partial_i | i = 1, 2, 3\}.$$

As shorthand notations, we use Z to denote an arbitrary vector field from \mathcal{Z} ; similarly, we use Z_g and Z_b to denote vectors from \mathcal{Z}_g and \mathcal{Z}_b respectively. Geometrically, a good vector field Z_g is tangential to the outgoing light cone C_0 , but a bad vector field Z_b is transversal to C_0 .

1.4.2. A word on null forms. Recall that a quadratic form Q over \mathbb{R}^{3+1} is a *null form* if $Q(\xi, \xi) = 0$ for all null vector $\xi \in \mathbb{R}^{3+1}$. The space of null forms are spanned by the following seven forms: $Q_0(\xi, \eta) = g(\xi, \eta)$ and $Q_{\alpha\beta}(\xi, \eta) = \xi_\alpha\eta_\beta - \eta_\alpha\xi_\beta$ ($0 \leq \alpha, \beta \leq 3$). Given scalar functions ϕ, ψ and a null form $Q(\xi, \eta) = Q^{\alpha\beta}\xi_\alpha\eta_\beta$, we use $Q(\nabla\phi, \nabla\psi)$ as a shorthand for $Q(\nabla\phi, \nabla\psi) = Q^{\alpha\beta}\partial_\alpha\phi\partial_\beta\psi$.

For a (conformal) Killing vector field $Z \in \mathcal{Z}$, we have

$$ZQ(\nabla\phi, \nabla\psi) = Q(\nabla Z\phi, \nabla\psi) + Q(\nabla\phi, \nabla Z\psi) + \tilde{Q}(\nabla\phi, \nabla\psi), \quad (1.13)$$

where \tilde{Q} is a null form, which may or may not be Q .

In terms of the null pair (L, \underline{L}) , a null form Q satisfies the following pointwise estimates

$$|Q(\nabla\phi, \nabla\psi)| \lesssim |L\phi| |\underline{L}\psi| + |\underline{L}\phi| |L\psi| + |\nabla\phi| |\nabla\psi| + (|L\phi| + |\underline{L}\phi|) |\nabla\psi| + |\nabla\phi| (|L\psi| + |\underline{L}\psi|). \quad (1.14)$$

In particular, on the right hand side of the inequality, the term $|\underline{L}\phi|^2$ does not appear.

1.4.3. Main features of the proof. We discuss main difficulties of the problem and also the ideas to get around them.

- Largeness/Loss of δ^{-1} in the short pulse region.

In the short pulse region, if one differentiates ϕ in the \underline{L} direction, then the resulting function will be approximately δ^{-1} times as large as the initial functions. Schematically, we can regard \underline{L} as $\underline{L} \sim \delta^{-1}$. Similarly, $L \sim 1$ and $\nabla \sim 1$.

The large factor δ^{-1} maybe fatal to the energy estimates for nonlinear terms. The resolution of this difficulty is exactly the basic philosophy of null conditions: if one term behaves badly, say $|\underline{L}\phi| \sim \delta^{-\frac{1}{2}}$ in the nonlinearities, it must be coupled with the a good term, say $L\phi$ or $\nabla\phi$, which are both of size $\delta^{\frac{1}{2}}$. Their product will then be a term of size 1 which will be manageable in the proof.

- Relaxation in δ for the propagation estimates.

On the initial hypersurface Σ_1 , it is easy to see that the data satisfy $\|\underline{L}\phi\|_{L_{\Sigma_1}^\infty} \sim \delta^{-\frac{1}{2}}$ and $\|\nabla\phi\|_{L_{\Sigma_1}^\infty} \sim \delta^{\frac{1}{2}}$. Up to a correct decay factor in t , we hope the size of $\underline{L}\phi$ and $\nabla\phi$ measured in δ can be propagated for later times, i.e. $\|\underline{L}\phi\|_{L_{\Sigma_1}^\infty} \sim \delta^{-\frac{1}{2}}$ and $\|\nabla\phi\|_{L_{\Sigma_1}^\infty} \sim \delta^{\frac{1}{2}}$ should be always true. Recall that the proof will be based on energy estimates. If we use \underline{L} as a multiplier vector field and integrate in $D_{\underline{u},u}$ in the short pulse region, the energy on the left hand side of (1.12) is $\int_{\underline{C}_u} |\underline{L}\phi|^2 + \int_{C_u} |\nabla\phi|^2$. Therefore, the expected propagation estimates suggest that $\int_{\underline{C}_u} |\underline{L}\phi|^2 \lesssim 1$ and $\int_{C_u} |\nabla\phi|^2 \lesssim \delta$. Therefore, in view of the form of the energy, the disparity of the δ power for these two quantities only gives the desired bound for $\underline{L}\phi$, but not for $\nabla\phi$. This may lead to the failure of closing the bootstrap argument.

To get around this difficulty, we pretend that the amplitude of $\nabla\phi$ was worse than that suggested by the initial data. The purpose of this relaxation is to make the two terms in $\int_{\underline{C}_u} |\underline{L}\phi|^2 + \int_{C_u} |\nabla\phi|^2$ comparable. More specifically, we first prove that $\|\nabla\phi\|_{L^\infty(\Sigma_1)} \lesssim 1$ can be propagated by controlling $\int_{\underline{C}_u} |\underline{L}\phi|^2 + \int_{C_u} |\nabla\phi|^2$ in terms of initial energies. In this way we are able to close the bootstrap argument. Then we use the proved energy estimate to recover the estimate $\|\nabla\phi\|_{L^\infty(\Sigma_t)} \lesssim \delta^{1/4} t^{-2}$ by affording to lose one derivative.

- Relaxation in the decay factor in the short pulse region.

According to the decay rate of linear waves, one may expect the decay of ϕ or more precisely the derivatives of ϕ should be $\frac{1}{t}$ or $\frac{1}{u}$ in the short pulse region. This expected decay will cause a loss of $\log t$ in the energy estimates since we may need to integrate a factor of size $\frac{1}{t}$ coming from the nonlinear term.

The idea to get around this point is also to relax the decay rate a little bit. This is why we choose $X = |\underline{u}|^{1-\varepsilon_0} L$ as a multiplier vector field instead of using the standard vectorfield S . The $|\underline{u}|^{-\varepsilon_0}$ will be amplified to $|\underline{u}|^{-2\varepsilon_0}$ in the energy estimates due to the nonlinearity. Therefore, we can gain a little more decay relative to the relaxed decay. This is just enough to close the argument for the energy estimates.

- Smallness of the solution on C_δ .

This is precisely the question that we will answer in Step 2 of the proof. As we discussed, in the short pulse region, we expect $\underline{L} \sim \delta^{-1}$. In particular, we expect that, for all the bad vector fields Z_b , we also have $Z_b \sim \delta^{-1}$. Therefore, for a given n , the restriction of $Z_b^n \phi$ on C_δ may be of size $\delta^{\frac{1}{2}-n}$. This is by no means small.

The key point of the proof is the following observation: on the 2-sphere $S_{1-\delta,\delta}$, i.e. the initial sphere of C_δ , the data vanish completely since they are compactly supported on Σ_1 between $S_{1-\delta,\delta}$ and $S_{1,0}$. Therefore, even the bad derivatives of ϕ are small initially. To get the smallness of ϕ , we will integrate along null geodesics on C_δ to trace all the information back to the data. In this way, we can show that up to an error of size $\delta^{\frac{1}{2}}$, all derivatives of ϕ are comparable to their initial values.

1.4.4. *Comparison with the previous work* [10]. We now discuss the difference between the present work and the previous work [10].

- Cauchy data versus characteristic scattering data.

In the present work, we consider the Cauchy data given on Σ_1 satisfying (1.3) and (1.4). However (1.3), (1.4) do not give the a priori decay profile. To prove pointwise decay estimates for ϕ , besides the standard vectorfield \underline{L} , one has to choose $\underline{u}^\alpha L, \alpha \in (1/2, 1)$ as a multiplier vectorfield. Here the index α is chosen in such a way that it is enough to prove decay estimates but the decay rate is not too strong to prove.

This should be compared with the characteristic scattering data considered in [10] given at the past null infinity $C_{-\infty}$. More specifically, the data for ϕ in [10] has the following form:

$$\lim_{u \rightarrow -\infty} |u| \phi(u, \underline{u}, \theta) = \delta^{1/2} \psi_0 \left(\frac{\underline{u}}{\delta}, \theta \right) \quad (1.15)$$

Here $\psi_0 : (0, 1) \times \mathbb{S}^2 \rightarrow \mathbb{R}$ is a compactly supported smooth function. Note that the data (1.15) has the property

$$\phi(u, \underline{u}, \theta) \sim \frac{\delta^{1/2}}{|u|} \psi_0 \left(\frac{\underline{u}}{\delta}, \theta \right) + o \left(\frac{1}{|u|} \right), \quad u \rightarrow -\infty. \quad (1.16)$$

Therefore the decay profile $\frac{1}{|u|}$ for ϕ is given a priori, which simplifies the proof of pointwise decay estimates for ϕ and its derivatives. In fact, the authors in [10] use the standard vectorfields L and \underline{L} as the multipliers to prove the energy estimates.

- Compactly supported data versus non-compactly supported trace.

In [10] the characteristic data (1.15) given at the past null infinity is compactly supported in $u \in (0, \delta)$. After solving the characteristic problem all the way up to $t = -1$, the restriction of solution on Σ_{-1} gives the data for the time-reversed Cauchy problem. The Cauchy data given in this way is implicitly and can never be compactly supported. On the contrary, the Cauchy data in the present work is given directly and explicitly. Moreover, as it is shown in (1.5), the data can be compactly supported. On the other hand, compared to the characteristic data at the past null infinity, which is compactly supported, the trace of the solution at the future null infinity ($\underline{u} = \infty$) in the present work is not compactly supported.

1.4.5. Applications in physical problems. We would like to discuss further applications of our method to other wave type equations, especially Yang-Mills equations in gauge theory and Einstein equations in general relativity. For both systems of equations, there is no known result to derive global asymptotic behaviors for large data problem. Taking Yang-Mills equations as an example. Let F be the Yang-Mills field. We define two 1-forms by contracting with L and \underline{L} on $S_{u, \underline{u}}$: $\alpha_F = i_L F$, $\underline{\alpha}_F = i_{\underline{L}} F$. Since F is an Lie-algebra valued two forms (of dimension 6), the rest two components of F are denoted by ρ_F and σ_F . This four components $\alpha_F, \underline{\alpha}_F, \rho_F$ and σ_F consists of a complete decomposition of F by using null frames. To make connections to our method, we make the following correspondence:

$$\alpha_F \mapsto L\phi, \quad \underline{\alpha}_F \mapsto \underline{L}\phi, \quad (\rho_F, \sigma_F) \mapsto \nabla \phi.$$

By the correspondence, we can use the corresponding energy ansatz for the each component respectively. Since Yang-Mills equations also have null structures, we expect our method can prove the first asymptotic description large data problem. Similarly, we can also study Einstein equations in such a way (by virtue of harmonic coordinates). This will be a forthcoming paper.

1.4.6. *Outline of the paper.* The rest of the paper is organized as follows:

In Section 2, we establish *a priori* energy estimates for higher order derivatives of the solution in the short pulse region. As consequences, first of all, we can construct the solution in the short pulse region; Secondly, we can obtain a smallness estimate for the solution on C_δ , i.e. the inner boundary of the short pulse data region.

In Section 3, with a modified Klainerman-Sobolev inequality, we construct global solutions in the small data region.

2. SHORT PULSE REGION

The goal of the current section is to construct the solution ϕ in the short pulse region. The construction relies on *a priori* energy estimates. We assume that the solution ϕ exists on spacetime domain $D_{\underline{u}^*, u^*}$. This domain is inside the short pulse region, i.e. $u^* \in (0, \delta)$ and $\underline{u}^* \in (1 - u^*, +\infty)$.

We first introduce the energy norms. Let $u, u' \in (0, u^*)$ and $\underline{u}, \underline{u}' \in (1 - u^*, \underline{u}^*)$. Let $C_u^{u'}$ be the part of the cone C_u so that $1 - u^* \leq \underline{u} \leq \underline{u}'$ and let $\underline{C}_u^{\underline{u}'}$ be the part of the cone \underline{C}_u so that $0 \leq u \leq u'$. Whenever there is no confusion, we will use C_u and \underline{C}_u instead of C_u^u and \underline{C}_u^u . We use Σ_1 to denote the annulus region $\{(r, \theta) \mid 1 - 2\delta \leq r \leq 1\}$ on $\{t = 1\}$ in the current section.

For a given $k \in \mathbb{Z}_{\geq 0}$, we introduce the following homogeneous norms:

$$\begin{aligned} E_k(u, \underline{u}) &= \sum_{\substack{Z_g \in \mathcal{Z}_g, Z_b \in \mathcal{Z}_b \\ 0 \leq l \leq k}} \left(\delta^l \|\nabla Z_b^l Z_g^{k-l} \phi\|_{L^2(C_u^{\underline{u}'})} + \delta^{l-\frac{1}{2}} \|\underline{u}\|^{\frac{\alpha}{2}} L Z_b^l Z_g^{k-l} \phi\|_{L^2(C_u^{\underline{u}'})} \right), \\ \underline{E}_k(u, \underline{u}) &= \sum_{\substack{Z_g \in \mathcal{Z}_g, Z_b \in \mathcal{Z}_b \\ 0 \leq l \leq k}} \left(\delta^l \|\underline{L} Z_b^l Z_g^{k-l} \phi\|_{L^2(\underline{C}_u^u)} + \delta^{l-\frac{1}{2}} \|\underline{u}\|^{\frac{\alpha}{2}} \nabla Z_b^l Z_g^{k-l} \phi\|_{L^2(\underline{C}_u^u)} \right). \end{aligned}$$

We also introduce the inhomogeneous norms:

$$E_{\leq k}(u, \underline{u}) = \sum_{0 \leq j \leq k} E_j(u, \underline{u}), \quad \underline{E}_{\leq k}(u, \underline{u}) = \sum_{0 \leq j \leq k} \underline{E}_j(u, \underline{u}).$$

On the initial hypersurface Σ_1 , we introduce the following initial energy norms:

$$E_{\leq n}(\Sigma_1) = \sum_{\substack{Z_g \in \mathcal{Z}_g, Z_b \in \mathcal{Z}_b \\ 0 \leq l \leq k, 0 \leq k \leq n}} \delta^l \|\underline{L} Z_b^l Z_g^{k-l} \phi\|_{L^2(\Sigma_1)} + \delta^{l-\frac{1}{2}} \|\nabla Z_b^l Z_g^{k-l} \phi\|_{L^2(\Sigma_1)} + \delta^{l-1} \|L Z_b^l Z_g^{k-l} \phi\|_{L^2(\Sigma_1)}.$$

According to the behavior of ϕ_0 and ϕ_1 on Σ_1 as well as the properties of commutation vectorfields, using the arguments in [10] and [11], we have the following lemma:

Lemma 2.1. *For all δ , we have*

$$E_{\leq n-1}(\Sigma_1) \lesssim I_n. \quad (2.1)$$

Here I_n is a constant depending only on n .

2.1. Main a priori estimates. This subsection is the central part of the paper. The goal is to bound $E_{\leq 3}(u, \underline{u})$ on $D_{\underline{u}^*, u^*}$ where the solution ϕ is assumed to exist.

Proposition 2.2. *There exists $\delta_0 > 0$, so that for all $\delta < \delta_0$, for all $u \in (0, u^*)$ and $\underline{u} \in (1 - u^*, \underline{u}^*)$, we have*

$$E_{\leq 3}(u, \underline{u}) + \underline{E}_{\leq 3}(u, \underline{u}) \leq C(I_4), \quad (2.2)$$

where $C(I_4)$ is a constant depending only on I_4 .

The proof of the proposition is based on a standard bootstrap argument. On $D_{\underline{u}^*, u^*}$, since we assume that ϕ exists, there is a large constant M , so that

$$E_{\leq 3}(u, \underline{u}) + \underline{E}_{\leq 3}(u, \underline{u}) \lesssim M, \quad (2.3)$$

for all $u \in (0, u^*)$ and $\underline{u} \in (1 - u^*, \underline{u}^*)$. The large constant M may depend on ϕ itself. The purpose of the bootstrap argument is to show that, if δ is sufficiently small, then one can choose M in such a way that it depends only on I_4 . Hence, we obtain the proof of (2.2)

2.1.1. Preliminary estimates. The goal of this subsection is to use the bootstrap assumption (2.3) to get estimates on lower order derivatives of ϕ (up to second derivatives).

We first recall the Sobolev inequalities on $S_{\underline{u}, u}$, $\underline{C}_{\underline{u}}$ and C_u in the short pulse region. Recall that in the short pulse region, we have $|\underline{u}| \sim r$ provided δ is sufficiently small. Let ϕ be a smooth function.

On $S_{\underline{u}, u}$, we have

$$\|\phi\|_{L^\infty(S_{\underline{u}, u})} \lesssim |\underline{u}|^{-1/2} (\|\phi\|_{L^4(S_{\underline{u}, u})} + \|\Omega\phi\|_{L^4(S_{\underline{u}, u})}), \quad (2.4)$$

$$\|\phi\|_{L^4(S_{\underline{u}, u})} \lesssim |\underline{u}|^{-1/2} (\|\phi\|_{L^2(S_{\underline{u}, u})} + \|\Omega\phi\|_{L^2(S_{\underline{u}, u})}). \quad (2.5)$$

On $\underline{C}_{\underline{u}}$, if in addition we assume that $\phi \equiv 0$ on C_0 , we have

$$\|\phi\|_{L^2(S_{\underline{u}, u})} \lesssim \|\underline{L}\phi\|_{L^2(\underline{C}_{\underline{u}})}^{1/2} \|\phi\|_{L^2(\underline{C}_{\underline{u}})}^{1/2}, \quad (2.6)$$

$$\|\phi\|_{L^4(S_{\underline{u}, u})} \lesssim |\underline{u}|^{-\frac{1}{2}} \|\underline{L}\phi\|_{L^2(\underline{C}_{\underline{u}})}^{1/2} (\|\phi\|_{L^2(\underline{C}_{\underline{u}})}^{1/2} + \|\Omega\phi\|_{L^2(\underline{C}_{\underline{u}})}^{1/2}).$$

We remark that the assumption $\phi \equiv 0$ on C_0 will be always true when we apply the above inequalities in the short pulse region in the rest of the paper, since the solution ϕ of the main equations (1.2) (if it exists) vanishes to infinite order on C_0 .

If ϕ is supported in the annular region $\{(r, \theta) | 1 - \delta \leq r \leq 1\}$ on the initial Cauchy hypersurface Σ_1 , we have

$$\|\phi\|_{L^2(S_{1-u, u})} \lesssim \delta^{1/2} (\|\underline{L}\phi\|_{L^2(\Sigma_1)} + \|L\phi\|_{L^2(\Sigma_1)}). \quad (2.7)$$

For the proof of the above inequalities, we refer the reader to [2].

We also recall the Gronwall's inequality. Let $f(t)$ be a non-negative function defined on an interval I with initial point t_0 . If f satisfies

$$\frac{d}{dt} f \leq a \cdot f + b$$

with two non-negative functions $a, b \in L^1(I)$, then for all $t \in I$, we have

$$f(t) \leq e^{A(t)} (f(t_0) + \int_{t_0}^t e^{-A(\tau)} b(\tau) d\tau)$$

where $A(t) = \int_{t_0}^t a(\tau) d\tau$.

We start to derive estimates and we treat \underline{u} as a fixed constant. By virtue of null pair (L, \underline{L}) , we rewrite the main system of equations (1.2) as

$$-L\underline{L}\phi + \not\Delta\phi + \frac{1}{r}(L\phi - \underline{L}\phi) = Q(\nabla\phi, \nabla\phi). \quad (2.8)$$

We remark that, ϕ is now a \mathbb{R}^N -valued function and the norms used in the rest of the paper are with respect to a fixed inner product in \mathbb{R}^N . For example, the symbol $|L\phi|$ denotes $\sqrt{\sum_{I \leq N} (L\phi^I)^2}$.

We also need to commute derivatives with (2.8). Recall that, for all $Z \in \mathcal{Z}$ except for $Z = S$, we have $[\square, Z] = 0$. Indeed, we have $[\square, S] = 2\square$. Combining this remark with (1.13), for all $k \geq 0$, we can

commute k vectors $Z_1, Z_2, \dots, Z_k \in \mathcal{Z}$ with (1.2) to obtain a semilinear wave equation for $Z_1 Z_2 \dots Z_k \phi$. We use the shorthand notation $Z^k \phi$ to denote $Z_1 Z_2 \dots Z_k \phi$, therefore, we have

$$\square Z^k \phi = \sum_{p+q \leq k} Q(\nabla Z^p \phi, \nabla Z^q \phi). \quad (2.9)$$

We combine (2.4), (2.5), (2.6) and bootstrap assumption (2.3). We first have

$$\begin{aligned} \|\nabla \phi\|_{L^4(S_{\underline{u}, u})} &\lesssim |\underline{u}|^{-\frac{1}{2}} \|\underline{L} \nabla \phi\|_{L^2(\underline{C}_{\underline{u}})}^{\frac{1}{2}} (\|\nabla \phi\|_{L^2(\underline{C}_{\underline{u}})}^{\frac{1}{2}} + |\underline{u}|^{\frac{1}{2}} \|\nabla^2 \phi\|_{L^2(\underline{C}_{\underline{u}})}) \\ &\lesssim |\underline{u}|^{-\frac{1}{2}} (|\underline{u}|^{-1} M)^{\frac{1}{2}} ((\delta^{\frac{1}{2}} |\underline{u}|^{-\frac{\alpha}{2}} M)^{\frac{1}{2}} + |\underline{u}|^{\frac{1}{2}} (\delta^{\frac{1}{2}} |\underline{u}|^{-\frac{2+\alpha}{2}} M)^{\frac{1}{2}}). \end{aligned}$$

Hence,

$$\|\nabla \phi\|_{L^4(S_{\underline{u}, u})} \lesssim \delta^{\frac{1}{4}} |\underline{u}|^{-1-\frac{\alpha}{4}} M.$$

Similarly, since in the bootstrap assumption (2.3), we have assumed bounds on four derivatives on ϕ , we can repeat the above argument to derive

$$\|\nabla \Omega \phi\|_{L^4(S_{\underline{u}, u})} \lesssim \delta^{\frac{1}{4}} |\underline{u}|^{-1-\frac{\alpha}{4}} M, \quad (2.10)$$

and

$$\|\nabla \Omega^2 \phi\|_{L^4(S_{\underline{u}, u})} \lesssim \delta^{\frac{1}{4}} |\underline{u}|^{-1-\frac{\alpha}{4}} M. \quad (2.11)$$

Combining (2.1.1) and (2.10), Sobolev inequality implies

$$\begin{aligned} \|\nabla \phi\|_{L^\infty(S_{\underline{u}, u})} &\lesssim |\underline{u}|^{-\frac{1}{2}} (\|\nabla \phi\|_{L^4(S_{\underline{u}, u})} + \|\nabla \Omega \phi\|_{L^4(S_{\underline{u}, u})}) \\ &\lesssim \delta^{\frac{1}{4}} |\underline{u}|^{-\frac{3}{2}-\frac{\alpha}{4}} M. \end{aligned} \quad (2.12)$$

Similarly, we have

$$\|\nabla \Omega \phi\|_{L^\infty(S_{\underline{u}, u})} \lesssim \delta^{\frac{1}{4}} |\underline{u}|^{-\frac{3}{2}-\frac{\alpha}{4}} M. \quad (2.13)$$

By repeating the above argument, for $0 \leq l \leq k \leq 2$, we can also easily obtain

$$\|\nabla Z_b^l Z_g^{k-l} \phi\|_{L^4(S_{\underline{u}, u})} \lesssim \delta^{\frac{1}{4}-l} |\underline{u}|^{-1-\frac{\alpha}{4}} M.$$

and for $0 \leq l \leq k \leq 1$

$$\|\nabla Z_b^l Z_g^{k-l} \phi\|_{L^\infty(S_{\underline{u}, u})} \lesssim \delta^{\frac{1}{4}-l} |\underline{u}|^{-\frac{3}{2}-\frac{\alpha}{4}} M. \quad (2.14)$$

We turn to the bound of $L\phi$ in $L^\infty(S_{\underline{u}, u})$. Let $a = \frac{1}{r} |\underline{L}\phi| + |\nabla \phi|$ and $b = |\underline{\Delta}\phi| + \frac{1}{r} |\underline{L}\phi| + |\underline{L}\phi \nabla \phi| + |\nabla \phi|^2$, in view of (1.14), (2.8) yields

$$\underline{L}|L\phi| \lesssim a|L\phi| + b.$$

We would like to integrate this equation directly along \underline{L} to derive the pointwise bound on $L\phi$. Since $L\phi$ vanishes along C_0 , in view of Gronwall's inequality, it suffices to control $\|a\|_{L_u^1 L^\infty(S_{\underline{u}, u})}$ and $\|b\|_{L_u^1 L^\infty(S_{\underline{u}, u})}$. We only give the estimates on $|\underline{L}\phi|$ appearing in a and b . The others can be estimated directly from (2.14). According to Sobolev inequality, we have

$$\begin{aligned} \|\underline{L}\phi\|_{L_u^1 L^\infty(S_{\underline{u}, u})} &\lesssim |\underline{u}|^{-1} \sum_{0 \leq j \leq 2} \|\Omega^j \underline{L}\phi\|_{L_u^1 L^\infty(S_{\underline{u}, u})} \\ &\lesssim |\underline{u}|^{-1} \delta^{\frac{1}{2}} \sum_{0 \leq j \leq 2} \|\Omega^j \underline{L}\phi\|_{L_u^2 L^\infty(S_{\underline{u}, u})} \\ &\lesssim |\underline{u}|^{-1} \delta^{\frac{1}{2}} M. \end{aligned}$$

Finally, we can prove

$$\begin{aligned} \|a\|_{L_u^1 L^\infty(S_{\underline{u},u})} &\lesssim |\underline{u}|^{-1} \delta^{-\frac{1}{2}} M, \\ \|b\|_{L_u^1 L^\infty(S_{\underline{u},u})} &\lesssim |\underline{u}|^{-2} \delta^{-\frac{1}{2}} M. \end{aligned}$$

Therefore, Gronwall's inequality provides us the following estimates for $L\phi$:

$$\|L\phi\|_{L^\infty(S_{\underline{u},u})} \lesssim \delta^{1/2} |\underline{u}|^{-2} M. \quad (2.15)$$

By virtue of (2.9) (where $k \leq 2$), we can also bound $LZ_b^l Z_g^{k-l} \phi$ in $L^2(S_{\underline{u},u})$ in a similar way. Therefore, for $0 \leq l \leq k \leq 2$, we have

$$\|LZ_b^l Z_g^{k-l} \phi\|_{L^2(S_{\underline{u},u})} \lesssim \delta^{1/2-l} |\underline{u}|^{-1} M. \quad (2.16)$$

We turn to the $L^\infty(S_{\underline{u},u})$ estimates on $\underline{L}\phi$. We start with a computation of $L(\underline{u}^2(\underline{L}\phi)^2)$:

$$\begin{aligned} L(\underline{u}^2(\underline{L}\phi)^2) &= 2\underline{u}(\underline{L}\phi)^2 + 2\underline{u}^2(\underline{L}\phi)(L\underline{L}\phi) \\ &= 2\underline{u}(\underline{L}\phi)^2 + 2\underline{u}^2(\underline{L}\phi) \left[\Delta\phi + \frac{1}{r}(L\phi - \underline{L}\phi) + Q(\nabla\phi, \nabla\phi) \right] \\ &\lesssim |\underline{u}|^2 |\underline{L}\phi|^2 \left(\frac{2}{r} - \frac{2}{\underline{u}} + |L\phi| + |\nabla\phi| \right) + |\underline{u}|^2 |\underline{L}\phi| \left(|\Delta\phi| + \frac{1}{r}|L\phi| + |L\phi||\nabla\phi| + |\nabla\phi|^2 \right). \end{aligned}$$

We make the following important observation: in the short pulse region, $|\frac{2}{r} - \frac{2}{\underline{u}}| \lesssim \frac{\delta}{|\underline{u}|^2}$. Therefore, if we define $y = |\underline{u}||\underline{L}\phi|$, according to the estimate obtained so far, the previous computation yields

$$Ly^2 \lesssim \left(\frac{\delta}{|\underline{u}|^2} + \frac{\delta^{\frac{1}{4}}}{|\underline{u}|^2} M \right) y^2 + \frac{\delta^{\frac{1}{4}}}{|\underline{u}|^2} My.$$

We divide both sides of the equation by y , thus, we have

$$L(|\underline{u}||\underline{L}\phi|) \lesssim \left(\frac{\delta}{|\underline{u}|^2} + \frac{\delta^{\frac{1}{4}}}{|\underline{u}|^2} M \right) (|\underline{u}||\underline{L}\phi|) + \frac{\delta^{\frac{1}{4}}}{|\underline{u}|^2} M.$$

By integrating directly this equation, if δ is sufficiently small, we obtain

$$|\underline{u}||\underline{L}\phi|(\underline{u}, u, \theta) - C|1 - u||\underline{L}\phi|(1 - u, u, \theta)| \lesssim \delta^{\frac{1}{4}} M. \quad (2.17)$$

where the absolute constant C comes from the use of Gronwall's inequality. Therefore, according to (2.7) and Lemma 2.1, we finally obtain

$$\|\underline{L}\phi\|_{L^\infty(S_{\underline{u},u})} \lesssim \frac{\delta^{-\frac{1}{2}}}{|\underline{u}|} I_3 + \delta^{\frac{1}{4}} |\underline{u}|^{-1} M. \quad (2.18)$$

We remark that the derivation of (2.15) and (2.18) depends on not only on the bootstrap assumption (2.3) but also the main equation (1.2). We summarize the estimates derived so far as follows:

$$\begin{aligned} \|\nabla Z_b^l Z_g^{k-l} \phi\|_{L^4(S_{\underline{u},u})} &\lesssim \delta^{\frac{1}{4}-l} |\underline{u}|^{-1-\frac{\alpha}{4}} M, \quad 0 \leq l \leq k \leq 2, \\ \|\nabla\phi\|_{L^\infty(S_{\underline{u},u})} &\lesssim \delta^{\frac{1}{4}} |\underline{u}|^{-\frac{3}{2}-\frac{\alpha}{4}} M, \\ \|L\phi\|_{L^\infty(S_{\underline{u},u})} &\lesssim \delta^{\frac{1}{2}} |\underline{u}|^{-2} M, \\ \|LZ_b^l Z_g^{k-l} \phi\|_{L^2(S_{\underline{u},u})} &\lesssim \delta^{1/2-l} |\underline{u}|^{-1} M, \quad 0 \leq l \leq k \leq 2, \\ \|\underline{L}\phi\|_{L^\infty(S_{\underline{u},u})} &\lesssim \frac{\delta^{-\frac{1}{2}}}{|\underline{u}|} I_3 + \delta^{\frac{1}{4}} |\underline{u}|^{-1} M, \end{aligned} \quad (2.19)$$

Remark 2.3. The bootstrap assumptions (2.3) involve relaxed estimates for $\nabla\phi$. Roughly speaking, in (2.3) we expect the behavior of $\nabla\phi$ with respect to δ and \underline{u} is approximately $|\underline{u}|^{-1-\frac{\alpha}{2}}M$, i.e.

$$\|\nabla\phi\|_{L^\infty(S_{\underline{u},u})} \sim |\underline{u}|^{-1-\frac{\alpha}{2}} \cdot \delta^0 \cdot M.$$

However, the estimates on $\nabla\phi$ in (2.19) shows that, by affording two more derivatives (via Sobolev inequalities), we can improve the bound on $\nabla\phi$: we get an extra $\delta^{\frac{1}{4}}$ factor and an extra $\underline{u}^{-\frac{1}{2}+\frac{\alpha}{4}}$ decay factor, i.e.,

$$\|\nabla\phi\|_{L^\infty(S_{\underline{u},u})} \sim |\underline{u}|^{-\frac{3}{2}-\frac{\alpha}{4}} \cdot \delta^{\frac{1}{4}} \cdot M.$$

2.1.2. *Estimates on $E_{\leq 2}$ and $\underline{E}_{\leq 2}$.* Recall that for $Z \in \mathcal{Z}$ and $k \geq 0$, we have

$$\square Z^k \phi = \sum_{p+q \leq k} Q(\nabla Z^p \phi, \nabla Z^q \phi). \quad (2.20)$$

In Section (2.1.2), we fix $k \leq 2$. Let $l \leq k$ be the number of Z_b 's appearing in Z^k , i.e. $Z^k = Z_b^l Z_g^{k-l}$. We use the vector field method outlined in the introduction to estimate $E_{\leq 2}$ and $\underline{E}_{\leq 2}$.

In the fundamental energy identity (1.12) and (2.20), we replace ϕ by $Z^k \phi$ and take $X = \underline{L}$ to obtain

$$\begin{aligned} \int_{C_u} |\nabla Z^k \phi|^2 + \int_{\underline{C}_{\underline{u}}} |\underline{L} Z^k \phi|^2 &= \int_{\Sigma_1} |\nabla Z^k \phi|^2 + |\underline{L} Z^k \phi|^2 + \iint_{D_{\underline{u},u}} Q(\nabla Z^k \phi, \nabla \phi) \underline{L} Z^k \phi \\ &+ \sum_{\substack{p+q \leq k, \\ p < k, q < k}} \iint_{D_{\underline{u},u}} Q(\nabla Z^p \phi, \nabla Z^q \phi) \underline{L} Z^i \phi - \iint_{D_{\underline{u},u}} \frac{1}{r} \underline{L} Z^k \phi \cdot L Z^k \phi. \end{aligned}$$

We multiply both sides of the equation by δ^{2l} to normalize the contribution from the initial data to be close to 1, therefore, we obtain

$$\begin{aligned} \delta^{2l} \int_{C_u} |\nabla Z^k \phi|^2 + \delta^{2l} \int_{\underline{C}_{\underline{u}}} |\underline{L} Z^k \phi|^2 &\lesssim I_3^2 + \delta^{2l} \left| \iint_{D_{\underline{u},u}} Q(\nabla Z^k \phi, \nabla \phi) \underline{L} Z^k \phi \right| \\ &+ \sum_{\substack{p+q \leq k, \\ p < k, q < k}} \delta^{2l} \left| \iint_{D_{\underline{u},u}} Q(\nabla Z^p \phi, \nabla Z^q \phi) \underline{L} Z^i \phi \right| + \delta^{2l} \left| \iint_{D_{\underline{u},u}} \frac{1}{r} \underline{L} Z^k \phi \cdot L Z^k \phi \right|. \end{aligned} \quad (2.21)$$

We rewrite the right-hand side of the above inequality as

$$I_3^2 + S + T + W.$$

where S , T and W denote the three bulk integral terms in (2.21). We will bound S , T and W one by one.

We begin with S , by definition, S is bounded by the sum of the following integrals:

$$\begin{aligned} S_1 &= \delta^{2l} \iint_{D_{\underline{u},u}} (|L\phi| + |\nabla\phi|) |\underline{L} Z^k \phi|^2, \\ S_2 &= \delta^{2l} \iint_{D_{\underline{u},u}} |\underline{L}\phi| |L Z^k \phi| |\underline{L} Z^k \phi|, \\ S_3 &= \delta^{2l} \iint_{D_{\underline{u},u}} |\underline{L}\phi| |\nabla Z^k \phi| |\underline{L} Z^k \phi|, \\ S_4 &= \delta^{2l} \iint_{D_{\underline{u},u}} |\nabla\phi| |L Z^k \phi| |\underline{L} Z^k \phi|, \\ S_5 &= \delta^{2l} \iint_{D_{\underline{u},u}} (|L\phi| + |\nabla\phi|) |\nabla Z^k \phi| |\underline{L} Z^k \phi|. \end{aligned}$$

It suffices to bound the S_i 's one by one.

For S_1 , in view of the $L^\infty(S_{\underline{u},u})$ estimates on $L\phi$ and $\nabla\phi$, we have

$$\begin{aligned} S_1 &\leq \delta^{2l} \int_{1-u}^u \int_0^u (\|L\phi\|_{L^\infty(S_{\underline{u}',u'})} + \|\nabla\phi\|_{L^\infty(S_{\underline{u}',u'})}) \|\underline{L}Z^k\phi\|_{L^2(S_{\underline{u}',u'})}^2 du' d\underline{u}' \\ &\leq \delta^{2l} \int_{1-u}^u \int_0^u \delta^{\frac{1}{4}} |\underline{u}'|^{-\frac{6+\alpha}{4}} M \|\underline{L}Z^k\phi\|_{L^2(S_{\underline{u}',u'})}^2 du' d\underline{u}' \\ &\lesssim \int_{1-u}^u \delta^{\frac{1}{4}} |\underline{u}'|^{-\frac{6+\alpha}{4}} M (\delta^{2l} \|\underline{L}Z^k\phi\|_{L^2(\underline{C}_{\underline{u}'})}^2) d\underline{u}'. \end{aligned}$$

In view of the bootstrap assumption, we bound $\delta^{2l} \|\underline{L}Z^k\phi\|_{L^2(\underline{C}_{\underline{u}'})}^2$ by M^2 . After an integration over \underline{u}' on $[1-u, \underline{u}]$, we have

$$S_1 \lesssim \delta^{\frac{1}{4}} (|1-u|^{-\frac{2+\alpha}{4}} - |\underline{u}|^{-\frac{2+\alpha}{4}}) M^3.$$

Because of $u \in [0, \delta]$, for sufficiently small δ , we have

$$S_1 \lesssim \delta^{\frac{1}{4}} M^3. \quad (2.22)$$

For S_2 , we have

$$S_2 \lesssim \delta^{2l} \int_{1-u}^u \|LZ^k\phi\|_{L_u^\infty L^2(S_{\underline{u}',u})} \|\underline{L}\phi\|_{L_u^2 L^\infty(S_{\underline{u}',u})} \|\underline{L}Z^k\phi\|_{L^2(\underline{C}_{\underline{u}'})} d\underline{u}'.$$

According to the bootstrap assumption (2.3) and the estimates (2.19), we have $\|LZ^k\phi\|_{L_u^\infty L^2(S_{\underline{u}',u})} \lesssim \delta^{\frac{1}{2}-l} |\underline{u}|^{-1} M$, $\|\underline{L}\phi\|_{L_u^2 L^\infty(S_{\underline{u}',u})} \lesssim I_3 |\underline{u}|^{-1}$ and $\|\underline{L}Z^k\phi\|_{L^2(\underline{C}_{\underline{u}'})} \lesssim \delta^{-l} M$, therefore, we can conclude that

$$S_2 \lesssim \delta^{\frac{1}{2}} M^2. \quad (2.23)$$

For S_3 , we have

$$\begin{aligned} S_3 &\lesssim \delta^{2l} \int_{1-u}^u \|\nabla Z^k\phi\|_{L_u^\infty L^2(S_{\underline{u}',u})} \|\underline{L}\phi\|_{L_u^2 L^\infty(S_{\underline{u}',u})} \|\underline{L}Z^k\phi\|_{L^2(\underline{C}_{\underline{u}'})} d\underline{u}' \\ &\lesssim I_3 \cdot M \cdot \delta^l \int_{1-u}^u \|\nabla Z^k\phi\|_{L_u^\infty L^2(S_{\underline{u}',u})} |\underline{u}|^{-1} d\underline{u}'. \end{aligned}$$

According to the L^4 estimates on $\nabla Z^k\phi$ on $S_{\underline{u},u}$ in (2.19), we have $\|\nabla Z^k\phi\|_{L^2(S_{\underline{u},u})} \lesssim \delta^{\frac{1}{4}-l} |\underline{u}|^{-\frac{1}{2}-\alpha} M$, this leads to

$$S_3 \lesssim \delta^{\frac{1}{4}} M^2. \quad (2.24)$$

For S_4 , we can proceed exactly as for S_2 (we just replace the factor $\underline{L}\phi$ in S_2 by $\nabla\phi$), this gives

$$S_4 \lesssim \delta^{\frac{5}{4}} M^3. \quad (2.25)$$

For S_5 , we can proceed exactly as for S_3 (we just replace the factor $|\underline{L}\phi|$ in S_3 by $|\nabla\phi| + |L\phi|$), this gives

$$S_5 \lesssim \delta M^3. \quad (2.26)$$

We now estimate the second term in (2.21), i.e. the estimates on T . According to the structure of null forms, we have

$$\begin{aligned} T &= \sum_{\substack{p+q \leq k, \\ p < k, q < k}} \delta^{2l} \left| \iint_{D_{\underline{u}, u}} Q(\nabla Z^p \phi, \nabla Z^q \phi) \underline{L} Z^k \phi \right| \\ &\lesssim \sum_{\substack{p+q \leq k, \\ p < k, q < k}} \delta^{2l} \iint_{D_{\underline{u}, u}} |\partial Z^p \phi| |\partial_g Z^q \phi| |\underline{L} Z^k \phi|, \end{aligned}$$

where $\partial \in \{\nabla, \underline{L}\}$ and $\partial_g \in \{\nabla, L\}$. For each given term in the above summ, let l' and l'' be total numbers of bad commutator Z_b 's appearing in Z^p and Z^q respectively. We remark that $l' + l'' \leq l$. Since $q \leq 1$, we have

$$T \lesssim \delta^{2l} \sum_{\substack{p+q \leq k, \\ p < k, q < k}} \int_{1-u}^u \int_0^u \|\partial Z^p \phi\|_{L^4(S_{\underline{u}', u'})} \|\partial_g Z^q \phi\|_{L^4(S_{\underline{u}', u'})} \|\underline{L} Z^k \phi\|_{L^2(S_{\underline{u}', u'})} du' d\underline{u}'$$

By the second of (2.6), we have:

$$\|\partial_g Z^q \phi\|_{L^4(S_{\underline{u}', u'})} \lesssim \underline{u}'^{-1/2} \|\underline{L} \partial_g Z^q \phi\|_{L^2(\underline{C}_{\underline{u}})}^{1/2} \left(\|\partial_g Z^q \phi\|_{L^2(\underline{C}_{\underline{u}})} + \|\Omega \partial_g Z^q \phi\|_{L^2(\underline{C}_{\underline{u}})} \right)^{1/2} \quad (2.27)$$

Now by (3.2), we have schematically:

$$L \sim \frac{1}{\underline{u}} S + \frac{1}{\underline{u}} \frac{x^i}{r} \Omega_{0i}, \quad \nabla \sim \frac{1}{\underline{u}} \Omega_{ij}$$

which imply, for any smooth function f :

$$\underline{L} \partial_g f \sim \frac{1}{\underline{u}} \underline{L} Z_g f, \quad \Omega f \sim \underline{u} \nabla f, \quad \partial_g f \sim \frac{1}{\underline{u}} Z_g f$$

If $\partial_g = \nabla$, then the second factor on the right hand side of (2.27) is bounded through the bootstrap assumption (2.3) by:

$$\left(\|\nabla Z^q \phi\|_{L^2(\underline{C}_{\underline{u}})} + \|\nabla Z_g Z^q \phi\|_{L^2(\underline{C}_{\underline{u}})} \right)^{1/2} \lesssim \delta^{1/4-l''/2} M^{1/2}$$

If $\partial_g = L$, by virtue of (2.16) the first term in the parenthesis is bounded by:

$$\|L Z^q \phi\|_{L^2(\underline{C}_{\underline{u}})} \lesssim \delta^{1-l''} \underline{u}^{-1} M$$

while for the second term we have:

$$\|\Omega \partial_g Z^q \phi\|_{L^2(\underline{C}_{\underline{u}})} \lesssim \|\nabla Z_g Z^q \phi\|_{L^2(\underline{C}_{\underline{u}})} \lesssim \delta^{1/2-l''} M^{1/2}$$

These together with (2.27) imply:

$$\|\partial_g Z^q \phi\|_{L^4(S_{\underline{u}', u'})} \lesssim \delta^{1/4-l''} M \underline{u}'^{-1} \quad (2.28)$$

Therefore (2.5) implies:

$$\begin{aligned} T &\lesssim \delta^{1/4} M \sum_{\substack{p+q \leq k, \\ p < k, q < k}} \int_{1-u}^u \underline{u}'^{-3/2} \int_0^u \left(\delta^{l-l''} \|\partial Z^q Z_g^{q'} \phi\|_{L^2(S_{\underline{u}', u'})} \delta^l \|\underline{L} Z^k \phi\|_{L^2(S_{\underline{u}', u'})} \right) du' d\underline{u}' \\ &\lesssim \delta^{1/4} M \sum_{\substack{p+q \leq k, \\ p < k, q < k}} \int_{1-u}^u \underline{u}'^{-3/2} \left(\delta^{l-l''} \|\partial Z^q Z_g^{q'} \phi\|_{L^2(\underline{C}_{\underline{u}'})} \delta^l \|\underline{L} Z^k \phi\|_{L^2(\underline{C}_{\underline{u}'})} \right) d\underline{u}' \\ &\lesssim \delta^{1/4} M^3 \int_{1-u}^u \underline{u}'^{-3/2} d\underline{u}' \end{aligned}$$

where $|q'| \leq 1$. This eventually yields

$$T \lesssim \delta^{\frac{1}{4}} M^3. \quad (2.29)$$

It remains to bound the third term W in (2.21). It is similar to T_2 . We simply bound $LZ^k\phi$ and $\underline{L}Z^k\phi$ on $\underline{C}_{\underline{u}}$. Although the bound of $|LZ^p\phi|$ on $\underline{C}_{\underline{u}}$ is not directly from the bootstrap assumption, in view of (2.19) and the fact that $k \leq 2$, we can bound $LZ^k\phi$ first on $L^2(S_{\underline{u},u})$ and then on $L^2(\underline{C}_{\underline{u}})$. This leads to

$$W \lesssim \delta M^2. \quad (2.30)$$

By combining (2.21) with (2.22), (2.23), (2.24), (2.25), (2.26), (2.29) and (2.30), for sufficiently small δ , we obtain

$$\delta^{2l} \int_{C_u} |\nabla Z^k \phi|^2 + \delta^{2l} \int_{\underline{C}_{\underline{u}}} |\underline{L}Z^k \phi|^2 \lesssim I_3^2 + \delta^{\frac{1}{4}} M^3.$$

In other words, for all $0 \leq l \leq k \leq 2$, we have

$$\delta^l \|\nabla Z_b^l Z_g^{k-l} \phi\|_{L^2(C_u)} + \delta^l \|\underline{L}Z_b^l Z_g^{k-l} \phi\|_{L^2(\underline{C}_{\underline{u}})} \lesssim I_3 + \delta^{\frac{1}{8}} M^{\frac{3}{2}}. \quad (2.31)$$

In the fundamental energy identity (1.12) and (2.20), we replace ϕ by $Z^k\phi$ and take $X = \underline{u}^\alpha L$ to obtain

$$\begin{aligned} \int_{C_u} |\underline{u}|^\alpha |LZ^k \phi|^2 + \int_{\underline{C}_{\underline{u}}} |\underline{u}|^\alpha |\nabla Z^k \phi|^2 &= \int_{\Sigma_1} |\underline{u}|^\alpha (|\nabla Z^k \phi|^2 + |LZ^k \phi|^2) + \iint_{\mathcal{D}_{\underline{u},u}} |\underline{u}|^\alpha Q(\nabla Z^k \phi, \nabla \phi) LZ^k \phi \\ &+ \sum_{\substack{p+q \leq k, \\ p < k, q < k}} \iint_{\mathcal{D}_{\underline{u},u}} |\underline{u}|^\alpha Q(\nabla Z^p \phi, \nabla Z^q \phi) LZ^k \phi + \iint_{\mathcal{D}_{\underline{u},u}} \frac{|\underline{u}|^\alpha}{r} \underline{L}Z^k \phi \cdot LZ^k \phi \\ &- 2\alpha \iint_{\mathcal{D}_{\underline{u},u}} |\underline{u}|^{\alpha-1} |\nabla Z^k \phi|^2. \end{aligned}$$

We multiply both sides of the equation by δ^{2l-1} to renormalize the contribution from the initial data to be close to 1. We remark that this normalization is respect to the relaxed estimates on $\nabla \phi$. By dropping of the last negative term in the above equation, we obtain

$$\begin{aligned} \delta^{2l-1} \int_{C_u} |\underline{u}|^\alpha |LZ^k \phi|^2 + \delta^{2l-1} \int_{\underline{C}_{\underline{u}}} |\underline{u}|^\alpha |\nabla Z^k \phi|^2 &\lesssim I_3^2 + \delta^{2l-1} \left| \iint_{\mathcal{D}_{\underline{u},u}} |\underline{u}|^\alpha Q(\nabla Z^k \phi, \nabla \phi) LZ^k \phi \right| \\ &+ \sum_{\substack{p+q \leq k, \\ p < k, q < k}} \delta^{2l-1} \left| \iint_{\mathcal{D}_{\underline{u},u}} |\underline{u}|^\alpha Q(\nabla Z^p \phi, \nabla Z^q \phi) LZ^k \phi \right| + \delta^{2l-1} \left| \iint_{\mathcal{D}_{\underline{u},u}} \frac{|\underline{u}|^\alpha}{r} \underline{L}Z^k \phi \cdot LZ^k \phi \right|, \end{aligned} \quad (2.32)$$

We rewrite the right-hand side of the above inequality as

$$I_3^2 + S + T + W.$$

where S , T and W denote the three bulk integral terms in (2.32). We now bound S , T and W one by one.

We begin with S . According to the definition of S and the structure (1.14) for null forms, S is bounded by the sum of the the following terms:

$$\begin{aligned} S_1 &= \delta^{2l-1} \iint_{\mathcal{D}_{\underline{u}, u}} |\underline{u}|^\alpha (|\underline{L}\phi| + |\nabla\phi|) |LZ^k\phi|^2, \\ S_2 &= \delta^{2l-1} \iint_{\mathcal{D}_{\underline{u}, u}} |\underline{u}|^\alpha (|\nabla\phi| + |L\phi|) |\underline{L}Z^k\phi| |LZ^k\phi|, \\ S_3 &= \delta^{2l-1} \iint_{\mathcal{D}_{\underline{u}, u}} |\underline{u}|^\alpha (|L\phi| + |\nabla\phi|) |\nabla Z^k\phi| |LZ^k\phi|, \\ S_4 &= \delta^{2l-1} \iint_{\mathcal{D}_{\underline{u}, u}} |\underline{u}|^\alpha |\underline{L}\phi| |\nabla Z^k\phi| |LZ^k\phi|. \end{aligned}$$

The idea to bound the S_i 's are exactly the same as before. Roughly speaking, we bound all the first order derivative components of $\nabla\phi$ in $L^\infty(S_{\underline{u}, u})$.

For S_1 , we have

$$\begin{aligned} S_1 &\leq \delta^{2l-1} \iint_{\mathcal{D}_{\underline{u}, u}} |\underline{u}|^\alpha (|\underline{u}|^{-1} \delta^{-\frac{1}{2}} M) |LZ^k\phi|^2 \\ &\lesssim \delta^{-\frac{1}{2}} M \int_0^u \left(\delta^{2l-1} \int_{C_{u'}} |\underline{u}|^\alpha |LZ^k\phi|^2 \right) du'. \end{aligned}$$

According to the bootstrap assumption on $\delta^{l-\frac{1}{2}} \|LZ^k\phi\|_{L^2(C_u)}$, we obtain

$$S_1 \lesssim \delta^{\frac{1}{2}} M^3. \quad (2.33)$$

For S_2 , since $k \leq 2$, we use the bound on $LZ^k\phi$ on $S_{\underline{u}, u}$ to derive $\delta^l \|LZ^k\phi\|_{L^2(\underline{C}_{\underline{u}'})} \lesssim \delta |\underline{u}|^{-1} M$. Therefore, we can proceed as follows:

$$S_2 \leq \delta^{2l-1} \int_{1-u}^u |\underline{u}|^\alpha (|\underline{u}|^{-\frac{3}{2} - \frac{\alpha}{4}} \delta^{\frac{1}{4}} M) \|\underline{L}Z^k\phi\|_{L^2(\underline{C}_{\underline{u}'})} \|LZ^k\phi\|_{L^2(\underline{C}_{\underline{u}'})} d\underline{u}.$$

According to the bootstrap assumptions, we finally obtain

$$S_2 \lesssim \delta^{\frac{1}{4}} M^3. \quad (2.34)$$

The estimates on S_3 can be obtained in a similar way as S_2 : we simply replace $\underline{L}Z^k\phi$ by $\nabla Z^k\phi$ and proceed exactly the same as before. This gives

$$S_3 \lesssim \delta^{\frac{3}{4}} M^3. \quad (2.35)$$

For S_4 , we first make the following remark:

Remark 2.4. *It seems to be natural to derive the estimates by putting $\nabla Z^k\phi$ in the $L^2(\underline{C}_{\underline{u}'})$ norm. In fact, this does not work due to the fact that we have relaxed the estimates on the rotational directions. To illustrate the idea, we may proceed as follows:*

$$\begin{aligned} S_4 &\leq \delta^{2l-1} \int_{1-u}^u |\underline{u}|^\alpha (|\underline{u}|^{-1} \delta^{-\frac{1}{2}} I_3) \|\nabla Z^k\phi\|_{L^2(\underline{C}_{\underline{u}'})} \|LZ^k\phi\|_{L^2(\underline{C}_{\underline{u}'})} d\underline{u} \\ &\leq \delta^{-1} \int_{1-u}^u |\underline{u}|^{\frac{\alpha}{2}} (|\underline{u}|^{-1} \delta^{-\frac{1}{2}} I_3) (\delta^{\frac{1}{2}} M) (\delta |\underline{u}|^{-1} M) d\underline{u}' \\ &\lesssim M^2. \end{aligned}$$

This estimate is certainly not good since we do not have a δ (to some positive power) factor in front of the possibly large constant M .

At this point, we have to use the bootstrap assumptions on the fourth order derivatives of ϕ to improve the relaxed estimates on ∇ -direction.

The above remark suggests to put $\nabla Z^k \phi$ in $L^4(S_{\underline{u},u})$ norm to get an extra $\delta^{\frac{1}{4}}$ factor. In fact, we have

$$S_4 \leq \delta^{2l-1} \int_{1-u}^u |\underline{u}|^\alpha \|\underline{L}\phi\|_{L_u^2 L^4(S_{\underline{u}',u})} \|\nabla Z^k \phi\|_{L_u^\infty L^4(S_{\underline{u}',u})} \|LZ^k \phi\|_{L^2(\underline{C}_{\underline{u}'})} d\underline{u}'.$$

Since we have already derived estimates on $\|\underline{L}\phi\|_{L^4(S_{\underline{u},u})}$ and $\|LZ^k \phi\|_{L^2(S_{\underline{u},u})}$ ($k \leq 2$), a direct computation yields

$$S_4 \lesssim \delta^{\frac{1}{4}} M^2. \quad (2.36)$$

We turn to the estimates on T . According to the structure of null forms, we have

$$T \lesssim \delta^{2l-1} \sum_{\substack{p+q \leq k, \\ p < k, q < k}} \iint_{D_{\underline{u},u}} |\underline{u}|^\alpha |\partial Z^p \phi| |\partial_g Z^q \phi| |LZ^k \phi|,$$

where $\partial \in \{\nabla, \underline{L}\}$ and $\partial_g \in \{\nabla, L\}$.

Here we postpone the estimate for T until we estimate the top order energy $E_{\leq 3}(\underline{u}, u)$, because the estimates for T corresponding to $E_{\leq 2}$ and $E_{\leq 3}$ are identical. Instead, we just state the result:

$$T \lesssim \delta^{-1} \sum_{\substack{p+q \leq k, \\ p < k, q < k}} \int_0^u \delta^{2l-1} \|\underline{u}'^{\alpha/2} LZ^k \phi\|_{L^2(C_{u'})}^2 du' + I_3^4 \quad (2.37)$$

It remains to control $W = \delta^{2l-1} \iint_{D_{\underline{u},u}} \frac{|\underline{u}|^\alpha}{r} |LZ^k \phi| |\underline{L}Z^k \phi|$. We proceed as follows

$$\begin{aligned} W &\lesssim \delta^{2l-1} \iint_{D_{\underline{u},u}} (\delta^{-\frac{1}{2}} |\underline{u}|^{\frac{\alpha}{2}} |LZ^k \phi|) (\delta^{\frac{1}{2}} |\underline{u}|^{-\frac{2-\alpha}{2}} |\underline{L}Z^k \phi|) \\ &\stackrel{\text{Cauchy-Schwarz}}{\lesssim} \delta^{2l-1} \left(\iint_{D_{\underline{u},u}} \frac{\delta}{|\underline{u}'|^{2-\alpha}} |\underline{L}Z^k \phi|^2 + \iint_{D_{\underline{u},u}} \frac{1}{\delta} |\underline{u}'|^{\frac{\alpha}{2}} |LZ^k \phi|^2 \right) \\ &= \delta^{2l} \int_{1-u}^u \frac{1}{|\underline{u}'|^{2-\alpha}} \|\underline{L}Z^k \phi\|_{L^2(C_{u'})}^2 d\underline{u}' + \delta^{2l-2} \int_0^u \|\underline{u}|^{\frac{\alpha}{2}} LZ^k \phi\|_{L^2(C_{u'})}^2 du'. \end{aligned}$$

The first term in the last line has already been controlled in (2.31). In view of the fact that $\alpha < 1$ (this is crucial to make the first factor integrable in $\underline{u}!$), for sufficiently small δ , we obtain

$$W \lesssim I_3^2 + \delta^{-1} \int_0^u \delta^{2l-1} \|\underline{u}|^{\frac{\alpha}{2}} LZ^k \phi\|_{L^2(C_{u'})}^2 du'. \quad (2.38)$$

By combining (2.32) with (2.33), (2.34), (2.35), (2.36), (2.37) and (2.38), for sufficiently small δ , we obtain

$$\delta^{2l-1} \int_{C_u} |\underline{u}|^\alpha |LZ^k \phi|^2 + \delta^{2l-1} \int_{\underline{C}_{\underline{u}}} |\underline{u}|^\alpha |\nabla Z^k \phi|^2 \lesssim I_3^4 + \delta^{\frac{1}{4}} M^3 + \delta^{-1} \int_0^u \delta^{2l-1} \|\underline{u}|^{\frac{\alpha}{2}} LZ^k \phi\|_{L^2(C_{u'})}^2 du'$$

The last term on the right-hand side can be removed by the Gronwall's inequality. This finally proves that, for all $0 \leq l \leq k \leq 2$, we have

$$\delta^{l-\frac{1}{2}} \|\underline{u}|^{\frac{\alpha}{2}} LZ_b^l Z_g^{k-l} \phi\|_{L^2(C_u)} + \delta^{l-\frac{1}{2}} \|\nabla Z_b^l Z_g^{k-l} \phi\|_{L^2(\underline{C}_{\underline{u}})} \lesssim C(I_3) + \delta^{\frac{1}{8}} M^{\frac{3}{2}}. \quad (2.39)$$

The estimates (2.31) and (2.39) together implies

$$E_{\leq 2}(u, \underline{u}) + \underline{E}_{\leq 2}(u, \underline{u}) \leq C(I_3) + \delta^{\frac{1}{8}} M^{\frac{3}{2}}. \quad (2.40)$$

2.1.3. *Estimates on E_3 and \underline{E}_3 .* We take $k = 3$ in (2.20). Let $l \leq k$ be the number of Z_b 's appearing in Z^3 , i.e. $Z^3 = Z_b^l Z_g^{3-l}$. We take $Z^3\phi$ in the place of ϕ in (1.12) and take the multiplier $X = \underline{L}$, this yields

$$\begin{aligned} \int_{C_u} |\nabla Z^3\phi|^2 + \int_{\underline{C}_u} |\underline{L}Z^3\phi|^2 &= \int_{\Sigma_1} |\nabla Z^3\phi|^2 + |\underline{L}Z^3\phi|^2 + \iint_{D_{\underline{u},u}} Q(\nabla Z^3\phi, \nabla\phi) \underline{L}Z^3\phi \\ &+ \sum_{\substack{p+q \leq 3, \\ p < 3, q < 3}} \iint_{D_{\underline{u},u}} Q(\nabla Z^p\phi, \nabla Z^q\phi) \underline{L}Z^3\phi - \iint_{D_{\underline{u},u}} \frac{1}{r} \underline{L}Z^3\phi \cdot LZ^3\phi. \end{aligned}$$

After a renormalization in δ , we obtain

$$\begin{aligned} \delta^{2l} \int_{C_u} |\nabla Z^3\phi|^2 + \delta^{2l} \int_{\underline{C}_u} |\underline{L}Z^3\phi|^2 &\lesssim I_4^2 + \delta^{2l} \left| \iint_{D_{\underline{u},u}} Q(\nabla Z^3\phi, \nabla\phi) \underline{L}Z^3\phi \right| \\ &+ \sum_{\substack{p+q \leq 3, \\ p < 3, q < 3}} \delta^{2l} \left| \iint_{D_{\underline{u},u}} Q(\nabla Z^p\phi, \nabla Z^q\phi) \underline{L}Z^3\phi \right| + \delta^{2l} \left| \iint_{D_{\underline{u},u}} \frac{1}{r} \underline{L}Z^3\phi \cdot LZ^3\phi \right| \quad (2.41) \\ &= I_4^2 + S + T + W, \end{aligned}$$

where S , T and W denote the three bulk integral terms. We will bound S , T and W one by one.

We start with S . It can be bounded by the sum of the following terms:

$$\begin{aligned} S_1 &= \delta^{2l} \iint_{D_{\underline{u},u}} |\underline{L}\phi| (|LZ^3\phi| + |\nabla Z^3\phi|) |\underline{L}Z^3\phi|, \\ S_2 &= \delta^{2l} \iint_{D_{\underline{u},u}} |L\phi| (|\nabla Z^3\phi| + |\underline{L}Z^3\phi|) |\underline{L}Z^3\phi|, \\ S_3 &= \delta^{2l} \iint_{D_{\underline{u},u}} |\nabla\phi| (|LZ^3\phi| + |\underline{L}Z^3\phi| + |\nabla Z^3\phi|) |\underline{L}Z^3\phi|. \end{aligned}$$

For S_1 , according to the $L^\infty(S_{u,u})$ estimates on $\underline{L}\phi$, we have

$$\begin{aligned} S_1 &\lesssim \delta^{2l-\frac{1}{2}} \iint_{D_{\underline{u},u}} |\underline{u}|^{-1} |\nabla Z^3\phi| |\underline{L}Z^3\phi| + \delta^{2l-\frac{1}{2}} \iint_{D_{\underline{u},u}} |\underline{u}|^{-\frac{1+\alpha}{2}} \cdot (|\underline{u}|^{\frac{\alpha}{2}} LZ^3\phi) \cdot (|\underline{u}|^{-\frac{1}{2}} |\underline{L}Z^3\phi|) \\ &= S_{11} + S_{12}. \end{aligned}$$

For S_{11} , according to Cauchy-Schwarz inequality, we have

$$\begin{aligned} S_{11} &\lesssim \delta^{2l-1} \iint_{D_{\underline{u},u}} |\nabla Z^3\phi|^2 + \delta^{2l} \iint_{D_{\underline{u},u}} |\underline{u}|^{-2} |\underline{L}Z^3\phi|^2 \\ &\lesssim \int_0^u \frac{1}{\delta} \cdot \delta^{2l} \int_{C_{u'}} |\nabla Z^3\phi|^2 du' + \int_{1-u}^u |\underline{u}'|^{-2} \delta^{2l} \int_{\underline{C}_{\underline{u}}} |\underline{L}Z^3\phi|^2 d\underline{u}'. \end{aligned} \quad (2.42)$$

For S_{12} , we still use Cauchy-Schwarz inequality to derive

$$\begin{aligned} S_{12} &\lesssim \delta^{2l-1} \iint_{D_{\underline{u},u}} (|\underline{u}|^{\frac{\alpha}{2}} |\underline{L}Z^3\phi|)^2 + \delta^{2l} \iint_{D_{\underline{u},u}} |\underline{u}|^{-2-\alpha} |\underline{L}Z^3\phi|^2 \\ &\lesssim \int_0^u \frac{1}{\delta} \cdot \delta^{2l} \int_{C_{u'}} |\underline{L}Z^3\phi|^2 du' + \int_{1-u}^u |\underline{u}'|^{-2-\alpha} \delta^{2l} \int_{\underline{C}_{\underline{u}}} |\underline{L}Z^3\phi|^2 d\underline{u}' \\ &\lesssim \delta M^2 + \int_{1-u}^u |\underline{u}'|^{-2-\alpha} \delta^{2l} \int_{\underline{C}_{\underline{u}}} |\underline{L}Z^3\phi|^2 d\underline{u}' \end{aligned}$$

Therefore, we obtain

$$S_1 \lesssim \delta M^2 + \int_0^u \frac{1}{\delta} \cdot \delta^{2l} \int_{C_{u'}} |\nabla Z^3 \phi|^2 du' + \int_{1-u}^u |\underline{u}'|^{-2} \delta^{2l} \int_{\underline{C}_{\underline{u}}} |\underline{L} Z^3 \phi|^2 d\underline{u}'. \quad (2.43)$$

For S_2 , according to the $L^\infty(S_{\underline{u},u})$ estimates on $L\phi$, we have

$$\begin{aligned} S_2 &\lesssim \delta^{2l+\frac{1}{2}} M \iint_{\mathcal{D}_{\underline{u},u}} |\underline{u}|^{-2} |\nabla Z^3 \phi| |\underline{L} Z^3 \phi| + \delta^{2l+\frac{1}{2}} M \iint_{\mathcal{D}_{\underline{u},u}} |\underline{u}|^{-2} |\underline{L} Z^3 \phi|^2 \\ &= S_{21} + S_{22}. \end{aligned}$$

For S_{21} , since $1 \lesssim |\underline{u}|$, we have

$$\begin{aligned} S_{21} &\lesssim \delta^{2l} M \iint_{\mathcal{D}_{\underline{u},u}} |\nabla Z^3 \phi|^2 + \delta^{2l+1} M \iint_{\mathcal{D}_{\underline{u},u}} |\underline{u}|^{-2} |\underline{L} Z^3 \phi|^2 \\ &\lesssim \int_0^u \delta^{2l} M \int_{C_{u'}} |\nabla Z^3 \phi|^2 du' + \delta M \int_{1-u}^u |\underline{u}'|^{-2} \delta^{2l} \int_{\underline{C}_{\underline{u}}} |\underline{L} Z^3 \phi|^2 d\underline{u}' \\ &\lesssim \delta M^3. \end{aligned}$$

For S_{22} , we have

$$\begin{aligned} S_{22} &\lesssim \delta^{\frac{1}{2}} M \int_{1-u}^u |\underline{u}'|^{-2} \delta^{2l} \int_{\underline{C}_{\underline{u}}} |\underline{L} Z^3 \phi|^2 d\underline{u}' \\ &\lesssim \delta^{\frac{1}{2}} M^3. \end{aligned}$$

Therefore, we obtain

$$S_2 \lesssim \delta^{\frac{1}{2}} M^3. \quad (2.44)$$

For S_3 , according to the $L^\infty(S_{\underline{u},u})$ estimates on $\nabla \phi$, it is bounded by the following three terms:

$$\begin{aligned} S_{31} &= \delta^{2l+\frac{1}{4}} M \iint_{\mathcal{D}_{\underline{u},u}} |\underline{u}|^{-\frac{3}{2}-\frac{\alpha}{4}} |\underline{L} Z^3 \phi| |\underline{L} Z^3 \phi|, \\ S_{32} &= \delta^{2l+\frac{1}{4}} M \iint_{\mathcal{D}_{\underline{u},u}} |\underline{u}|^{-\frac{3}{2}-\frac{\alpha}{4}} |\underline{L} Z^3 \phi|^2, \\ S_{33} &= \delta^{2l+\frac{1}{4}} M \iint_{\mathcal{D}_{\underline{u},u}} |\underline{u}|^{-\frac{3}{2}-\frac{\alpha}{4}} |\nabla Z^3 \phi| |\underline{L} Z^3 \phi|. \end{aligned}$$

To bound S_{31} , we follow exactly the same way for S_{11} , this yields

$$S_{31} \lesssim \delta^{\frac{1}{4}} M^3.$$

To bound S_{32} , we follow exactly the same way for S_{22} , this yields

$$S_{32} \lesssim \delta^{\frac{1}{4}} M^3.$$

To bound S_{33} , we follow exactly the same way for S_{21} , this yields

$$S_{33} \lesssim \delta^{\frac{1}{4}} M^3.$$

Therefore, we obtain

$$S_3 \lesssim \delta^{\frac{1}{4}} M^3. \quad (2.45)$$

We turn to the estimates on T . According to the structure of null forms, we have

$$T \lesssim \sum_{\substack{p+q \leq 3, \\ p \leq 2, q \leq 2}} \delta^{2l} \iint_{\mathcal{D}_{\underline{u},u}} |\partial Z^p \phi| |\partial_g Z^q \phi| |\underline{L} Z^3 \phi|.$$

where $\partial \in \{\nabla, \underline{L}\}$ and $\partial_g \in \{\nabla, L\}$. By using exactly the same method as we derive (2.29), we obtain

$$T \lesssim \delta^{\frac{1}{4}} M^3 \quad (2.46)$$

It remains to bound $W = \delta^{2l} \iint_{\mathcal{D}_{\underline{u}, u}} \frac{1}{r} |\underline{L}Z^3\phi| |\underline{L}Z^3\phi|$. According to Cauchy-Schwarz inequality, we have

$$\begin{aligned} W &\lesssim \delta^{2l} \left(\iint_{\mathcal{D}_{\underline{u}, u}} \frac{\delta}{|\underline{u}'|^{2-\alpha}} |\underline{L}Z^3\phi|^2 + \iint_{\mathcal{D}_{\underline{u}, u}} \frac{1}{\delta} |\underline{u}'|^{\frac{\alpha}{2}} |\underline{L}Z^3\phi|^2 \right) \\ &= \int_{1-u}^u \frac{1}{|\underline{u}'|^{2-\alpha}} \cdot \delta^{2l+1} \|\underline{L}Z^3\phi\|_{L^2(\underline{C}_{\underline{u}'})}^2 d\underline{u}' + \delta^{2l-1} \int_0^u \|\underline{u}|^{\frac{\alpha}{2}} \underline{L}Z^3\phi\|_{L^2(C_{u'})}^2 du'. \end{aligned}$$

This yields $W \lesssim \delta M^2$. Combining this estimate with (2.43), (2.44), (2.45) and (2.46), we obtain

$$\delta^{2l} \int_{C_u} |\nabla Z^3\phi|^2 + \delta^{2l} \int_{\underline{C}_{\underline{u}}} |\underline{L}Z^3\phi|^2 \lesssim I_4^2 + \delta^{\frac{1}{4}} M^3.$$

In other words, for all $0 \leq l \leq 3$, we have

$$\delta^l \|\nabla Z_b^l Z_g^{3-l}\phi\|_{L^2(C_u)} + \delta^l \|\underline{L}Z_b^l Z_g^{3-l}\phi\|_{L^2(\underline{C}_{\underline{u}})} \lesssim I_4 + \delta^{\frac{1}{8}} M^{\frac{3}{2}}. \quad (2.47)$$

Similar to the derivation for (2.32) (by taking $k = 3$), we have

$$\begin{aligned} &\delta^{2l-1} \int_{C_u} |\underline{u}|^\alpha |\underline{L}Z^3\phi|^2 + \delta^{2l-1} \int_{\underline{C}_{\underline{u}}} |\underline{u}|^\alpha |\nabla Z^3\phi|^2 \lesssim I_4^2 + \delta^{2l-1} \left| \iint_{\mathcal{D}_{\underline{u}, u}} |\underline{u}|^\alpha Q(\nabla Z^3\phi, \nabla\phi) \underline{L}Z^3\phi \right| \\ &\quad + \sum_{\substack{p+q \leq 3, \\ p \leq 2, q \leq 2}} \delta^{2l-1} \left| \iint_{D_{\underline{u}, u}} |\underline{u}|^\alpha Q(\nabla Z^p\phi, \nabla Z^q\phi) \underline{L}Z^3\phi \right| + \delta^{2l-2} \left| \iint_{D_{\underline{u}, u}} \frac{|\underline{u}|^\alpha}{r} \underline{L}Z^3\phi \cdot \underline{L}Z^3\phi \right|, \end{aligned}$$

We rewrite the above inequality as

$$\delta^{2l-1} \int_{C_u} |\underline{u}|^\alpha |\underline{L}Z^3\phi|^2 + \delta^{2l-1} \int_{\underline{C}_{\underline{u}}} |\underline{u}|^\alpha |\nabla Z^3\phi|^2 \lesssim I_4^2 + S + T + W. \quad (2.48)$$

where S , T and W denote the three bulk integral terms in an obvious way. We now bound S , T and W one by one.

We begin with S which is bounded by the sum of the the following terms:

$$\begin{aligned} S_1 &= \delta^{2l-1} \iint_{\mathcal{D}_{\underline{u}, u}} |\underline{u}|^\alpha (|\underline{L}\phi| + |\nabla\phi|) |\underline{L}Z^3\phi|^2, \\ S_2 &= \delta^{2l-1} \iint_{\mathcal{D}_{\underline{u}, u}} |\underline{u}|^\alpha (|\nabla\phi| + |\underline{L}\phi|) |\underline{L}Z^3\phi| |\underline{L}Z^3\phi|, \\ S_3 &= \delta^{2l-1} \iint_{\mathcal{D}_{\underline{u}, u}} |\underline{u}|^\alpha (|\underline{L}\phi| + |\nabla\phi|) |\nabla Z^3\phi| |\underline{L}Z^3\phi|, \\ S_4 &= \delta^{2l-1} \iint_{\mathcal{D}_{\underline{u}, u}} |\underline{u}|^\alpha |\underline{L}\phi| |\nabla Z^3\phi| |\underline{L}Z^3\phi|. \end{aligned}$$

For S_1 , we have

$$\begin{aligned} S_1 &\leq \delta^{2l-1} \iint_{\mathcal{D}_{\underline{u}, u}} |\underline{u}|^\alpha (|\underline{u}|^{-1} \delta^{-\frac{1}{2}} M) |\underline{L}Z^3\phi|^2 \\ &\lesssim \delta^{-\frac{1}{2}} M \int_0^u \left(\delta^{2l-1} \int_{C_{u'}} |\underline{u}|^\alpha |\underline{L}Z^k\phi|^2 \right) du' \\ &\lesssim \delta^{\frac{1}{2}} M^3. \end{aligned}$$

For S_2 , we use L^∞ bound on $\nabla\phi$ and $L\phi$. Since we have already derived estimates on $E_{\leq 2}(u, \underline{u})$ and $\underline{E}_{\leq 2}(u, \underline{u})$, for sufficiently small δ , we indeed have

$$|L\phi| + |\nabla\phi| \lesssim |\underline{u}|^{-\frac{3}{2}-\frac{\alpha}{4}} \delta^{\frac{1}{4}} C(I_3). \quad (2.49)$$

We remark that this estimate is better than those in (2.19) since we have improved the big bootstrap constant M to be a constant depending only on the size I_n of the rescaled data. Therefore, according to Cauchy-Schwarz inequality, we have

$$\begin{aligned} S_2 &\lesssim C(I_3) \delta^{2l-1} \iint_{\mathcal{D}_{\underline{u}, u}} (|\underline{u}|^{-\frac{3}{2}-\frac{\alpha}{4}} \delta^{\frac{1}{4}}) |\underline{L}Z^3\phi| |LZ^3\phi| \\ &\lesssim \delta^{\frac{1}{4}} \int_{1-u}^u \frac{1}{|\underline{u}'|^{3+\frac{3}{2}\alpha}} \cdot \delta^{2l} \|\underline{L}Z^3\phi\|_{L^2(\underline{C}_{u'})}^2 du' + \delta^{\frac{1}{4}} \int_0^u \frac{1}{\delta} \cdot \delta^{2l-1} \|\underline{u}\|^{\frac{\alpha}{2}} \|\underline{L}Z^3\phi\|_{L^2(C_{u'})}^2 du' \\ &\lesssim \delta^{\frac{1}{4}} M^2. \end{aligned}$$

For S_3 , we have

$$\begin{aligned} S_3 &\lesssim \delta^{2l-1} \iint_{\mathcal{D}_{\underline{u}, u}} (|\underline{u}|^{-\frac{3}{2}-\frac{\alpha}{4}} \delta^{\frac{1}{4}} M) |\nabla Z^3\phi| |LZ^3\phi| \\ &\lesssim M \delta^{-\frac{1}{4}} \int_0^u (\delta^l \|\nabla Z^3\phi\|_{L^2(C_{u'})}) (\delta^{l-\frac{1}{2}} \|\underline{u}\|^{\frac{\alpha}{2}} \|\underline{L}Z^3\phi\|_{L^2(C_{u'})}) du' \\ &\lesssim \delta^{\frac{3}{4}} M^3. \end{aligned}$$

For S_4 , we have

$$\begin{aligned} S_4 &\lesssim \delta^{2l-1} \iint_{\mathcal{D}_{\underline{u}, u}} (|\underline{u}|^{-1+\alpha} \delta^{-\frac{1}{2}} I_3) |\nabla Z^3\phi| |LZ^3\phi| \\ &\lesssim \delta^{-1} \int_0^u (\delta^l \|\nabla Z^3\phi\|_{L^2(C_{u'})}) (\delta^{l-\frac{1}{2}} \|\underline{u}\|^{\frac{\alpha}{2}} \|\underline{L}Z^3\phi\|_{L^2(C_{u'})}) du'. \end{aligned}$$

By virtue of (2.47), we can bound $\delta^l \|\nabla Z^3\phi\|_{L^2(C_{u'})}$ to derive

$$S_4 \lesssim C(I_4) M.$$

The estimates on S_1 , S_2 , S_3 and S_4 together yield

$$S \lesssim \delta^{\frac{1}{4}} M^3 + C(I_4) M. \quad (2.50)$$

For T , according to (1.14), we have

$$T \lesssim \sum_{\substack{p+q \leq 3, \\ p \leq 2, q \leq 2}} \delta^{2l-1} \iint_{\mathcal{D}_{\underline{u}, u}} |\underline{u}|^\alpha |\partial Z^p\phi| |\partial_g Z^q\phi| |LZ^3\phi|.$$

where $\partial \in \{\nabla, \underline{L}\}$ and $\partial_g \in \{\nabla, \underline{L}\}$.

We first consider $\partial_g = \nabla, \partial = \underline{L}$, and denote its contribution by T_1 , then we see all the other cases are lower order compared to this case. By (2.5) we have:

$$T_1 \lesssim \delta^{2l-1} \sum_{\substack{p+q \leq 3, \\ p \leq 2, q \leq 2}} \int_{1-u}^u \int_0^u \underline{u}'^{\alpha/2} \|\underline{L}Z^p\phi\|_{L^4(S_{u', \underline{u}'})} \|\nabla Z^q\phi\|_{L^4(S_{u', \underline{u}'})} \|\underline{u}'^{\alpha/2} LZ^3\phi\|_{L^2(S_{u', \underline{u}'})} du' d\underline{u}'$$

By the second of (2.6),

$$\begin{aligned} \delta^{l'} \|\underline{L}Z^p\phi\|_{L^4(S_{\underline{u}', \underline{u}'}')} &\lesssim \delta^{l'} \underline{u}'^{-1/2} \|\underline{L}Z_b Z^p\phi\|_{L^2(\underline{C}_{\underline{u}'})}^{1/2} \\ &\quad \left(\|\underline{L}Z^p\phi\|_{L^2(\underline{C}_{\underline{u}'})} + \|\underline{L}Z_g Z^p\phi\|_{L^2(\underline{C}_{\underline{u}'})} \right)^{1/2} \\ &\lesssim \delta^{-1/2} \underline{u}'^{-1/2} \left(I_4 + \delta^{1/8} M^{3/2} \right) \lesssim \delta^{-1/2} \underline{u}'^{-1/2} I_4 \end{aligned}$$

provided that δ is sufficiently small.

On the other hand, by (2.5),

$$\delta^{l''} \|\nabla Z^q\phi\|_{L^4(S_{\underline{u}', u'}')} \lesssim \underline{u}'^{-1/2} \left(\delta^{l''} \|\nabla Z^q\phi\|_{L^2(S_{\underline{u}', u'}')} + \delta^{l''} \|\nabla Z_g Z^q\phi\|_{L^2(S_{\underline{u}', u'}')} \right)$$

Therefore we have:

$$\begin{aligned} T_1 &\lesssim \delta^{l-1} \sum_{\substack{p+q \leq 3, \\ p \leq 2, q \leq 2}} \int_0^u \delta^{-1/2} I_4 \left(\delta^{l''} \|\nabla Z^q\phi\|_{L^2(C_{u'})} + \delta^{l''} \|\nabla Z_g Z^q\phi\|_{L^2(C_{u'})} \right) \\ &\quad \cdot \|\underline{u}'^{\alpha/2} LZ^3\phi\|_{L^2(C_{u'})} du' \lesssim \delta^{l-1} \sum_{\substack{p+q \leq 3, \\ p \leq 2, q \leq 2}} \int_0^u \delta^{-1/2} I_4^2 \|\underline{u}'^{\alpha/2} LZ^3\phi\|_{L^2(C_{u'})} du' \end{aligned}$$

By Cauchy-Schwarz, this implies:

$$T_1 \lesssim \delta^{-1} \sum_{\substack{p+q \leq 3, \\ p \leq 2, q \leq 2}} \int_0^u \delta^{2l-1} \|\underline{u}'^{\alpha/2} LZ^3\phi\|_{L^2(C_{u'})}^2 du' + I_4^4$$

If $\partial = \underline{L}$, $\partial_g = L$, we denote its contribution by T_2 , then by the estimates we have derived for $\|\underline{u}'^{\alpha/2} LZ^q\phi\|_{L^2(C_u)}$, a similar argument leads to the estimate on T_2 :

$$T_2 \lesssim \delta^{-1/2} \sum_{\substack{p+q \leq 3, \\ p \leq 2, q \leq 2}} \int_0^u \delta^{2l-1} \|\underline{u}'^{\alpha/2} LZ^3\phi\|_{L^2(C_{u'})}^2 du' + \delta^{1/2} I_4^4$$

If $\partial = \nabla$, the estimates for $\partial_g Z^q\phi$ are the same as before. While for $\nabla Z^p\phi$, we have, if δ is sufficiently small:

$$\begin{aligned} \delta^{l''} \|\underline{u}'^{\alpha/2} \nabla Z^p\phi\|_{L^4(S_{u', \underline{u}'})} &\lesssim \delta^{l''} \underline{u}'^{-1+\alpha/4} \|\underline{L}Z^p Z_g\phi\|_{L^2(\underline{C}_{\underline{u}'})}^{1/2} \\ &\quad \cdot \left(\|\underline{u}'^{\alpha/2} \nabla Z^p\phi\|_{L^2(\underline{C}_{\underline{u}'})} + \|\underline{u}'^{\alpha/2} \nabla Z_g Z^p\phi\|_{L^2(\underline{C}_{\underline{u}'})} \right)^{1/2} \\ &\lesssim \delta^{1/4} I_4 \end{aligned}$$

This bound is better than that of $\delta^{l''} \|\underline{L}Z^p\phi\|_{L^4(S_{u', \underline{u}'})}$. Therefore finally we obtain:

$$T \lesssim \delta^{-1} \sum_{\substack{p+q \leq 3, \\ p \leq 2, q \leq 2}} \int_0^u \delta^{2l-1} \|\underline{u}'^{\alpha/2} LZ^3\phi\|_{L^2(C_{u'})}^2 du' + I_4^4 \quad (2.51)$$

It remains to control $W = \delta^{2l-1} \iint_{\mathcal{D}_{\underline{u}, u}} \frac{|\underline{u}|^\alpha}{r} |LZ^3\phi| |\underline{L}Z^3\phi|$. By Cauchy-Schwarz inequality, we have

$$\begin{aligned} W &\lesssim \delta^{2l-1} \left(\iint_{\mathcal{D}_{\underline{u}, u}} \frac{\delta}{|\underline{u}'|^{2-\alpha}} |LZ^3\phi|^2 + \iint_{\mathcal{D}_{\underline{u}, u}} \frac{1}{\delta} |\underline{u}'|^{\frac{\alpha}{2}} |LZ^3\phi|^2 \right) \\ &= \delta^{2l} \int_{1-u}^u \frac{1}{|\underline{u}'|^{2-\alpha}} \|\underline{L}Z^3\phi\|_{L^2(\underline{C}_{\underline{u}'})}^2 du' + \delta^{2l-2} \int_0^u \|\underline{u}'^{\frac{\alpha}{2}} LZ^3\phi\|_{L^2(C_{u'})}^2 du'. \end{aligned}$$

We use (2.47) to bound the first term in the last line. Since $\alpha < 1$, for sufficiently small δ , we obtain

$$W \lesssim I_4^2 + \delta^{-1} \int_0^u \delta^{2l-1} \|\underline{u}\|^{\frac{\alpha}{2}} LZ^3 \phi\|_{L^2(C_{u'})}^2 du'. \quad (2.52)$$

By combining (2.50), (2.51) and (2.52), we obtain

$$\begin{aligned} & \delta^{2l-1} \int_{C_u} |\underline{u}|^\alpha |LZ^3 \phi|^2 + \delta^{2l-1} \int_{\underline{C}_u} |\underline{u}|^\alpha |\nabla Z^3 \phi|^2 \\ & \lesssim I_4^4 + \delta^{\frac{1}{4}} M^3 + C(I_3)M + \delta^{-1} \int_0^u \delta^{2l-1} \|\underline{u}\|^{\frac{\alpha}{2}} LZ^3 \phi\|_{L^2(C_{u'})}^2 du' \end{aligned}$$

The last term on the right-hand side can be removed by the Gronwall's inequality so that

$$\begin{aligned} & \delta^{2l-1} \int_{C_u} |\underline{u}|^\alpha |LZ^3 \phi|^2 + \delta^{2l-1} \int_{\underline{C}_u} |\underline{u}|^\alpha |\nabla Z^3 \phi|^2 \\ & \lesssim I_4^4 + \delta^{\frac{1}{4}} M^3 + C(I_3)M. \end{aligned}$$

This finally proves that, for all $0 \leq l \leq 3$, we have

$$\delta^{l-\frac{1}{2}} \|\underline{u}\|^{\frac{\alpha}{2}} LZ_b^l Z_g^{3-l} \phi\|_{L^2(C_u)} + \delta^{l-\frac{1}{2}} \|\nabla Z_b^l Z_g^{3-l} \phi\|_{L^2(\underline{C}_u)} \lesssim C(I_4) + C(I_3)M^{\frac{1}{2}} + \delta^{\frac{1}{8}} M^{\frac{3}{2}}. \quad (2.53)$$

By combining this estimates with (2.53) and (2.40), for sufficiently small δ , this finally proves

$$E_{\leq 3}(u, \underline{u}) + \underline{E}_{\leq 3}(u, \underline{u}) \leq C(I_4). \quad (2.54)$$

This is the end of the bootstrap argument and the Proposition 2.2 has been proved.

2.2. Higher Order Estimates. This subsection is devoted to prove a higher order analogue of Proposition 2.2:

Proposition 2.5. *Given a positive integer $n \geq 3$, there exists $\delta_0 > 0$, so that for all $\delta < \delta_0$, for all $u \in (0, u^*)$ and $\underline{u} \in (1 - u^*, \underline{u}^*)$, we have*

$$E_{\leq n}(u, \underline{u}) + \underline{E}_{\leq n}(u, \underline{u}) \leq C(I_{n+1}), \quad (2.55)$$

and

$$\begin{aligned} & \|\nabla Z_b^l Z_g^{k-l} \phi\|_{L^\infty(S_{\underline{u}, u})} \lesssim \delta^{\frac{1}{4}-l} |\underline{u}|^{-\frac{3}{2}-\frac{\alpha}{4}} C(I_{n+1}), \quad 0 \leq l \leq k \leq n-3, \\ & \|LZ_b^l Z_g^{k-l} \phi\|_{L^\infty(S_{\underline{u}, u})} \lesssim \delta^{\frac{1}{2}-l} |\underline{u}|^{-2} C(I_{n+1}), \quad 0 \leq l \leq k \leq n-3, \\ & \|\underline{L}Z_b^l Z_g^{k-l} \phi\|_{L^\infty(S_{\underline{u}, u})} \lesssim \delta^{-\frac{1}{2}-l} |\underline{u}|^{-1} C(I_{n+1}), \quad 0 \leq l \leq k \leq n-3. \end{aligned} \quad (2.56)$$

where $C(I_{n+1})$ is a constant depending only on I_{n+1} .

Remark 2.6. Although $C(I_n)$ and δ_0 in the proposition may depend on the integer n , in the rest of the paper, we only need the result for $n = 12$.

We prove (2.55) and (2.56) together by induction on n . For $n = 3$, the proposition has been achieved in the previous subsection. For $n \geq 4$, we assume that the proposition holds for all n' so that $n' \leq n-1$. To prove for n , we first make the following bootstrap assumption: We choose a large constant M , so that

$$E_n(u, \underline{u}) + \underline{E}_n(u, \underline{u}) \lesssim M, \quad (2.57)$$

for all $u \in (0, u^*)$ and $\underline{u} \in (1 - u^*, \underline{u}^*)$. We remark that M may depend on ϕ at the moment. We will show that, if δ is sufficiently small, then we can make M depend only on I_{n+1} . We also remark that the induction hypothesis is

$$E_{\leq n-1}(u, \underline{u}) + \underline{E}_{\leq n-1}(u, \underline{u}) \lesssim C(I_n), \quad (2.58)$$

and

$$\begin{aligned} \|\nabla Z_b^l Z_g^{k-l} \phi\|_{L^\infty(S_{\underline{u},u})} &\lesssim \delta^{\frac{1}{4}-l} |\underline{u}|^{-\frac{3}{2}-\frac{\alpha}{4}} C(I_n), \quad 0 \leq l \leq k \leq n-4, \\ \|L Z_b^l Z_g^{k-l} \phi\|_{L^\infty(S_{\underline{u},u})} &\lesssim \delta^{\frac{1}{2}-l} |\underline{u}|^{-2} C(I_n), \quad 0 \leq l \leq k \leq n-4, \\ \|\underline{L} Z_b^l Z_g^{k-l} \phi\|_{L^\infty(S_{\underline{u},u})} &\lesssim \delta^{-\frac{1}{2}-l} |\underline{u}|^{-1} C(I_n), \quad 0 \leq l \leq k \leq n-4. \end{aligned} \quad (2.59)$$

for all $u \in (0, u^*)$ and $\underline{u} \in (1 - u^*, \underline{u}^*)$.

We claim that, together with the induction hypothesis (2.58) and (2.59), the bootstrap assumption (2.57) implies

$$\begin{aligned} \|\nabla Z_b^l Z_g^{n-2-l} \phi\|_{L^\infty(S_{\underline{u},u})} &\lesssim \delta^{\frac{1}{4}-l} |\underline{u}|^{-\frac{3}{2}-\frac{\alpha}{4}} M, \quad 0 \leq l \leq n-3, \\ \|L Z_b^l Z_g^{n-2-l} \phi\|_{L^\infty(S_{\underline{u},u})} &\lesssim \delta^{\frac{1}{2}-l} |\underline{u}|^{-2} M, \quad 0 \leq l \leq n-3, \\ \|\underline{L} Z_b^l Z_g^{n-2-l} \phi\|_{L^\infty(S_{\underline{u},u})} &\lesssim \delta^{-\frac{1}{2}-l} |\underline{u}|^{-1} M, \quad 0 \leq l \leq n-3. \end{aligned} \quad (2.60)$$

The bound on $\|\nabla Z_b^l Z_g^{n-3-l} \phi\|_{L^\infty(S_{\underline{u},u})}$ is straightforward: we simply use Sobolev inequalities by affording two more Ω_{ij} derivatives. The derivation is exactly the same as for (2.12).

The bound on $\|L Z_b^l Z_g^{n-3-l} \phi\|_{L^\infty(S_{\underline{u},u})}$ relies on the (2.9), i.e.

$$\square Z^{n-3} \phi = \sum_{p+q \leq n-3} Q(\nabla Z^p \phi, \nabla Z^q \phi). \quad (2.61)$$

According to the structure of null forms, we can rewrite it as the following inequality:

$$\underline{L}|L Z^{n-3} \phi| \lesssim a|L Z^{n-3} \phi| + b, \quad (2.62)$$

where

$$a = \frac{1}{r} + (|\underline{L}\phi| + |\nabla\phi|),$$

and

$$b = |\underline{L} Z^{n-3} \phi| + \frac{1}{r} |\underline{L} Z^{n-3} \phi| + \sum_{p+q \leq n-3} (|\underline{L} Z^p \phi| |\nabla Z^q \phi| + |\nabla Z^p \phi| |\nabla Z^q \phi|).$$

We claim that

$$\begin{aligned} \|a\|_{L_u^1 L^\infty(S_{\underline{u},u})} &\lesssim |\underline{u}|^{-1} \delta^{-\frac{1}{2}-l} M, \\ \|b\|_{L_u^1 L^\infty(S_{\underline{u},u})} &\lesssim |\underline{u}|^{-2} \delta^{-\frac{1}{2}-l} M. \end{aligned} \quad (2.63)$$

To prove this claim, we first notice that all the terms have already been bounded by the induction hypothesis except for the top order terms, i.e. $|\underline{L} Z^{n-3} \phi|$, $\frac{1}{r} |\underline{L} Z^{n-3} \phi|$, $|\underline{L} Z^{n-3} \phi| |\nabla \phi|$, $|\underline{L} \phi| |\nabla Z^{n-3} \phi|$ and $|\nabla \phi| |\nabla Z^{n-3} \phi|$ appeared in b . In view of the bound on $\|\nabla Z_b^l Z_g^{n-3-l} \phi\|_{L^\infty(S_{\underline{u},u})}$ derived above, it suffices to bound $|\underline{L} Z^{n-3} \phi|$. According to Sobolev inequality, we have

$$\begin{aligned} \|\underline{L} Z^{n-3} \phi\|_{L_u^1 L^\infty(S_{\underline{u},u})} &\lesssim |\underline{u}|^{-1} \sum_{0 \leq j \leq 2} \|\Omega^j \underline{L} Z^{n-3} \phi\|_{L_u^1 L^2(S_{\underline{u},u})} \\ &\lesssim |\underline{u}|^{-1} \delta^{\frac{1}{2}} \sum_{0 \leq j \leq 2} \|\underline{L} \Omega^j Z^{n-3} \phi\|_{L_u^2 L^2(S_{\underline{u},u})} \\ &\lesssim |\underline{u}|^{-1} \delta^{\frac{1}{2}} M. \end{aligned}$$

We thus proved (2.63). By virtue of Gronwall's inequality, (2.62) yields the desired estimates for $L Z_b^l Z_g^{k-l} \phi$ in (2.60).

The estimates on $\|\underline{L} Z_b^l Z_g^{n-3-l} \phi\|_{L^\infty(S_{\underline{u},u})}$ relies on the use of equation (2.20). In fact, we have

$$\square Z^{n-3} \phi = \sum_{p+q \leq n-3} Q(\nabla Z_g^p \phi, \nabla Z_g^q \phi).$$

Let $y = |\underline{u}| |\underline{L} Z_b^l Z_g^{n-3-l} \phi|$. By computing $L(\underline{u}^2 (\underline{L} Z_b^l Z_g^{n-3-l} \phi)^2)$, we have

$$Ly^2 \lesssim \left(\frac{\delta}{|\underline{u}|^2} + \frac{\delta^{\frac{1}{4}}}{|\underline{u}|^2} M \right) y^2 + \frac{\delta^{\frac{1}{4}-l}}{|\underline{u}|^2} My.$$

By integrating this equation, we obtain

$$|\underline{u}| |\underline{L} Z_b^l Z_g^{n-3-l} \phi|(\underline{u}, u, \theta) - C|1 - u| |\underline{L} Z_b^l Z_g^{n-3-l} \phi|(1 - u, u, \theta) \lesssim \delta^{-\frac{1}{4}-l} M, \quad (2.64)$$

Therefore, according to (2.7), we finally obtain

$$\|\underline{L} Z_b^l Z_g^{n-3-l} \phi\|_{L^\infty(S_{\underline{u}, u})} \lesssim \frac{\delta^{-\frac{1}{2}-l}}{|\underline{u}|} I_{n+1} + \delta^{-\frac{1}{4}-l} |\underline{u}|^{-1} M.$$

To finish the proof of Proposition 2.55, it remains to improve the constant M in (2.57). The procedure is exactly the same as for the proof of $E_3(u, \underline{u})$ and $\underline{E}_3(u, \underline{u})$ in previous subsection.

We replace ϕ by $Z^n \phi$ and take $X = \underline{L}$ in (1.12) and (2.20), this yields

$$\begin{aligned} \delta^{2l} \int_{C_u} |\nabla Z^n \phi|^2 + \delta^{2l} \int_{\underline{C}_{\underline{u}}} |\underline{L} Z^n \phi|^2 &\lesssim I_{n+1}^2 + \delta^{2l} \left| \iint_{D_{\underline{u}, u}} Q(\nabla Z^n \phi, \nabla \phi) \underline{L} Z^n \phi \right| \\ &+ \sum_{\substack{p+q \leq n, \\ p < n, q < n}} \delta^{2l} \left| \iint_{D_{\underline{u}, u}} Q(\nabla Z^p \phi, \nabla Z^q \phi) \underline{L} Z^n \phi \right| + \delta^{2l} \left| \iint_{D_{\underline{u}, u}} \frac{1}{r} \underline{L} Z^n \phi \cdot L Z^n \phi \right|. \end{aligned} \quad (2.65)$$

We rewrite the right-hand side $I_{n+1}^2 + S + T + W$, where S , T and W denote the three bulk integral terms in (2.65). We will bound S , T and W one by one.

We start with S which bounded by the sum of the following terms:

$$\begin{aligned} S_1 &= \delta^{2l} \iint_{D_{\underline{u}, u}} |\underline{L} \phi| (|L Z^n \phi| + |\nabla Z^n \phi|) |\underline{L} Z^n \phi|, \\ S_2 &= \delta^{2l} \iint_{D_{\underline{u}, u}} |L \phi| (|\nabla Z^n \phi| + |\underline{L} Z^n \phi|) |\underline{L} Z^n \phi|, \\ S_3 &= \delta^{2l} \iint_{D_{\underline{u}, u}} |\nabla \phi| (|L Z^n \phi| + |\underline{L} Z^n \phi| + |\nabla Z^n \phi|) |\underline{L} Z^n \phi|. \end{aligned}$$

In view of the forms of S_1 , S_2 and S_3 appeared in the subsection for the estimates on $E_3(u, \underline{u})$ and $\underline{E}_3(u, \underline{u})$, i.e. the derivation of the inequalities (2.43), (2.44) and (2.45), we can proceed *exactly* in the same way (simply replace all the $Z^3 \phi$ by $Z^n \phi$). We take S_1 as an example to illustrate the process: by the bound on $\underline{L} \phi$ in $L^\infty(S_{\underline{u}, u})$, we have

$$\begin{aligned} S_1 &\lesssim \delta^{2l-\frac{1}{2}} \iint_{D_{\underline{u}, u}} |\underline{u}|^{-1} |\nabla Z^n \phi| |\underline{L} Z^n \phi| + \delta^{2l-\frac{1}{2}} \iint_{D_{\underline{u}, u}} |\underline{u}|^{-\frac{1+\alpha}{2}} \cdot (|\underline{u}|^{\frac{\alpha}{2}} L Z^n \phi) \cdot (|\underline{u}|^{-\frac{1}{2}} |\underline{L} Z^n \phi|) \\ &= S_{11} + S_{12}. \end{aligned}$$

We bound S_{11} exactly as the derivation for (2.42):

$$\begin{aligned} S_{11} &\lesssim \delta^{2l-1} \iint_{D_{\underline{u}, u}} |\nabla Z^n \phi|^2 + \delta^{2l} \iint_{D_{\underline{u}, u}} |\underline{u}|^{-2} |\underline{L} \Omega^3 \phi|^2 \\ &\lesssim \int_0^u \frac{1}{\delta} \cdot \delta^{2l} \int_{C_{u'}} |\nabla Z^n \phi|^2 du' + \int_{1-u}^u |\underline{u}'|^{-2} \delta^{2l} \int_{\underline{C}_{\underline{u}}} |\underline{L} Z^n \phi|^2 d\underline{u}'. \end{aligned}$$

Similarly, we have

$$S_{12} \lesssim \delta^{\frac{1}{4}} M^2.$$

We give the final result on as follows:

$$S \lesssim \delta^{\frac{1}{4}} M^2 + \int_0^u \frac{1}{\delta} \cdot \delta^{2l} \int_{C_{u'}} |\nabla Z^n \phi|^2 du' + \int_{1-u}^u |\underline{u}'|^{-2} \delta^{2l} \int_{\underline{C}_{\underline{u}}} |\underline{L} Z^n \phi|^2 d\underline{u}'. \quad (2.66)$$

For T , we have

$$T \lesssim \sum_{\substack{p+q \leq n, \\ p \leq n-1, q \leq n-1}} \delta^{2l} \iint_{D_{\underline{u}, u}} |\partial Z^p \phi| |\partial_g Z^q \phi| |\underline{L} Z^n \phi|.$$

where $\partial \in \{\nabla, \underline{L}\}$ and $\partial_g \in \{\nabla, L\}$. The estimate for T follows exactly the same as we derive (2.29). we have:

$$T \lesssim \delta^{1/4} M^3 \quad (2.67)$$

For $W = \delta^{2l} \iint_{D_{\underline{u}, u}} \frac{1}{r} |L Z^n \phi| |\underline{L} Z^n \phi|$, we have

$$\begin{aligned} W &\lesssim \int_{1-u}^u \frac{1}{|\underline{u}'|^{2-\alpha}} \cdot \delta^{2l+1} \|\underline{L} Z^n \phi\|_{L^2(\underline{C}_{\underline{u}})}^2 d\underline{u}' + \delta^{2l-1} \int_0^u \| |\underline{u}|^{\frac{\alpha}{2}} L Z^n \phi \|_{L^2(C_{u'})}^2 du' \\ &\lesssim \delta M^2. \end{aligned}$$

The estimates on S , T and W , together with (2.65), imply that

$$\begin{aligned} \delta^{2l} \int_{C_u} |\nabla Z^n \phi|^2 + \delta^{2l} \int_{\underline{C}_{\underline{u}}} |\underline{L} Z^n \phi|^2 &\lesssim I_{n+1}^2 + \delta^{\frac{1}{4}} M^3 \\ &\quad + \int_0^u \frac{1}{\delta} \cdot \delta^{2l} \int_{C_{u'}} |\nabla Z^n \phi|^2 du' + \int_{1-u}^u |\underline{u}'|^{-2} \delta^{2l} \int_{\underline{C}_{\underline{u}}} |\underline{L} Z^n \phi|^2 d\underline{u}'. \end{aligned}$$

The last two terms can be removed by Gronwall's inequality. Therefore, we obtain

$$\delta^l \|\nabla Z_b^l Z_g^{n-l} \phi\|_{L^2(C_u)} + \delta^l \|\underline{L} Z_b^l Z_g^{n-l} \phi\|_{L^2(\underline{C}_{\underline{u}})} \lesssim I_{n+1} + \delta^{\frac{1}{8}} M^{\frac{3}{2}}. \quad (2.68)$$

We now change the multiplier vector field to $u^\alpha L$ to derive

$$\begin{aligned} \delta^{2l-1} \int_{C_u} |\underline{u}|^\alpha |L Z^n \phi|^2 + \delta^{2l-1} \int_{\underline{C}_{\underline{u}}} |\underline{u}|^\alpha |\nabla Z^n \phi|^2 &\lesssim I_{n+1}^2 + \delta^{2l-1} \left| \iint_{D_{\underline{u}, u}} |\underline{u}|^\alpha Q(\nabla Z^n \phi, \nabla \phi) L Z^n \phi \right| \\ &\quad + \sum_{\substack{p+q \leq n, \\ p \leq n-1, q \leq n-1}} \delta^{2l-1} \left| \iint_{D_{\underline{u}, u}} |\underline{u}|^\alpha Q(\nabla Z^p \phi, \nabla Z^q \phi) L Z^n \phi \right| + \delta^{2l-2} \left| \iint_{D_{\underline{u}, u}} \frac{|\underline{u}|^\alpha}{r} \underline{L} Z^n \phi \cdot L Z^n \phi \right|, \end{aligned}$$

We rewrite the above inequality as

$$\delta^{2l-1} \int_{C_u} |\underline{u}|^\alpha |L Z^n \phi|^2 + \delta^{2l-1} \int_{\underline{C}_{\underline{u}}} |\underline{u}|^\alpha |\nabla Z^n \phi|^2 \lesssim I_{n+1}^2 + S + T + W. \quad (2.69)$$

where S , T and W denote the three bulk integral terms. We bound S , T and W one by one.

To bound S , we can follow exactly the same way as the derivation for (2.50) (we simply replace all the $Z^3 \phi$'s by $Z^n \phi$), this gives

$$S \lesssim \delta^{\frac{1}{4}} M^3 + C(I_{n+1})M.$$

To bound T , we can follow exactly the same way as the derivation for (2.51) (we simply replace all the $Z^3\phi$'s by $Z^n\phi$ and $Z^2\phi$'s by $Z^{n-1}\phi$). We then obtain

$$T \lesssim \delta^{-1} \sum_{\substack{p+q \leq n, \\ p \leq n-1, q \leq n-1}} \int_0^u \delta^{2l-1} \|\underline{u}'^{\alpha/2} LZ^n\phi\|_{L^2(C_{u'})}^2 du' + I_{n+1}^4$$

To bound W , we can follow exactly the same way as the derivation for (2.52) by replacing the $Z^3\phi$'s by $Z^n\phi$, this gives

$$W \lesssim I_{n+1}^2 + \delta^{-1} \int_0^u \delta^{2l-1} \|\underline{u}\|^{\frac{\alpha}{2}} LZ^n\phi\|_{L^2(C_{u'})}^2 du'.$$

The estimates on S , T and W , together with (2.65), imply that

$$\begin{aligned} & \delta^{2l-1} \int_{C_u} |\underline{u}|^\alpha |LZ^n\phi|^2 + \delta^{2l-1} \int_{\underline{C}_u} |\underline{u}|^\alpha |\nabla Z^n\phi|^2 \\ & \lesssim I_{n+1}^4 + \delta^{\frac{1}{4}} M^3 + C(I_n)M + \delta^{-1} \int_0^u \delta^{2l-1} \|\underline{u}\|^{\frac{\alpha}{2}} LZ^n\phi\|_{L^2(C_{u'})}^2 du' \end{aligned}$$

By the Gronwall's inequality again, for $l \leq n$, we finally obtain

$$\delta^{l-\frac{1}{2}} \|\underline{u}\|^{\frac{\alpha}{2}} LZ_b^l Z_g^{n-l}\phi\|_{L^2(C_u)} + \delta^{l-\frac{1}{2}} \|\nabla Z_b^l Z_g^{n-l}\phi\|_{L^2(\underline{C}_u)} \lesssim C(I_{n+1}) + C(I_n)M^{\frac{1}{2}} + \delta^{\frac{1}{8}} M^{\frac{3}{2}}. \quad (2.70)$$

For sufficiently small δ , the estimate (2.68) and (2.70) show that

$$E_n(u, \underline{u}) + \underline{E}_n(u, \underline{u}) \leq C(I_{n+1}).$$

This completes the bootstrap argument and the Proposition 2.5 has been proved.

2.3. Existence based on a priori estimates. The existence of solutions of (1.2) follows immediately from the a priori energy estimates derived previously. Since the procedure is standard, we only give a sketch of the proof in this subsection.

We start with solving local solution for Cauchy problem with data prescribed on Σ_1 with $1 - \delta \leq r \leq 1$. Therefore, we obtain a local solution confined in the region bounded by C_δ and \underline{C}_1 . In particular, on a neighborhood of $S_{1,0}$ on the incoming cone \underline{C}_1 , the solution has been constructed.

We then use C_0 and \underline{C}_1 as initial hypersurfaces. The classical local existence result [9] of Rendall can be applied in this situation. Therefore, we know that there exists a solution in the entire spacetime neighborhood (which lies in the domain of dependence of C_0 and \underline{C}_1) of $S_{1,0}$. Combined with the local solution of the Cauchy problem, we have constructed a local solution for $t \in [1, 1 + \epsilon]$ for some small ϵ .

Since the a priori energy estimates (as well as the accompanying L^∞ estimates) depends only on the size I_n of the rescaled data on Σ_1 , this solution is well behaved on $\Sigma_{1+\epsilon}$. Therefore, we can use this as initial surface (instead of Σ_1) to repeat the above argument. Eventually, we obtain an solution in the entire short pulse region.

2.4. Improved Estimates on C_δ . Recall that, given $n \geq 12$, in the short pulse region, we have derived the following a priori L^∞ estimates on the solution ϕ :

$$\begin{aligned} \|\nabla Z_b^l Z_g^{k-l}\phi\|_{L^\infty(S_{\underline{u},u})} & \lesssim \delta^{\frac{1}{4}-l} |\underline{u}|^{-\frac{3}{2}-\frac{\alpha}{4}} C(I_{n+1}), \quad 0 \leq l \leq k \leq n-3, \\ \|L Z_b^l Z_g^{k-l}\phi\|_{L^\infty(S_{\underline{u},u})} & \lesssim \delta^{\frac{1}{2}-l} |\underline{u}|^{-2} C(I_{n+1}), \quad 0 \leq l \leq k \leq n-3, \\ \|\underline{L} Z_b^l Z_g^{k-l}\phi\|_{L^\infty(S_{\underline{u},u})} & \lesssim \delta^{-\frac{1}{2}-l} |\underline{u}|^{-1} C(I_{n+1}), \quad 0 \leq l \leq k \leq n-3. \end{aligned}$$

The goal of this section is to improve these bounds for the solution on C_δ . More precisely, we will prove that

Proposition 2.7. *On C_δ , for sufficiently small δ , we have*

$$\begin{aligned} \|\nabla Z_b^l Z_g^{k-l} \phi\|_{L^\infty(S_{\underline{u}, \delta})} &\lesssim \delta^{\frac{1}{4}} |\underline{u}|^{-\frac{3}{2} - \frac{\alpha}{4}} C(I_{n+1}), \quad 0 \leq l \leq k \leq n-3, \\ \|L Z_b^l Z_g^{k-l} \phi\|_{L^\infty(S_{\underline{u}, \delta})} &\lesssim \delta^{\frac{1}{4}} |\underline{u}|^{-2} C(I_{n+1}), \quad 0 \leq l \leq k \leq n-3, \\ \|\underline{L} Z_b^l Z_g^{k-l} \phi\|_{L^\infty(S_{\underline{u}, \delta})} &\lesssim \delta^{\frac{1}{4}} |\underline{u}|^{-1} C(I_{n+1}), \quad 0 \leq l \leq k \leq n-3. \end{aligned} \quad (2.71)$$

Notice the power of δ for $L Z_b^l Z_g^{k-l} \phi$ has been modified to $\frac{1}{2}$. Since C_0 is also the outer boundary of the small data region (region I), the smallness (in terms of δ) of the solution stated in the proposition is indispensable for the construction of a global solution in the small data region. As we mentioned in the introduction, the proof relies on the following observation: on the $S_{1-\delta, \delta}$ or equivalently the lower boundary of C_δ , the data are *identically zero*. This is because that the data are compactly supported on Σ_1 between $S_{1-\delta, \delta}$ and $S_{1,0}$. Therefore, even for bad derivatives of ϕ , it is small at least initially. The idea of the proof is to integrate along the L direction to show that the smallness indeed propagates.

We use the induction argument on the pair (l, k) ($0 \leq k \leq n-3, 0 \leq l \leq k$) to prove (2.71). First of all, we give an order on the set of such pairs: we say that $(l', k') < (l, k)$ if one of the following holds: (1) $k' < k$ or (2) $l' < l, k = k'$. We do the induction with respect to this order.

For $(l, k) = (0, 0)$, the bounds on $L\phi$ and $\nabla\phi$ are clear. It remains to prove that

$$\|\underline{L}\phi\|_{L^\infty(S_{\underline{u}, \delta})} \lesssim \delta^{\frac{1}{4}} |\underline{u}|^{-1} C(I_n).$$

Recall that, in (2.17), we have obtained

$$|u| |\underline{L}\phi|(u, u, \theta) - C|u| |\underline{L}\phi|(1-u, u, \theta)| \lesssim \delta^{\frac{1}{4}} M.$$

In view of the higher order energy estimates derived in the previous subsection, the constant M should be replaced by $C(I_n)$. Let $u = \delta$, then the second term vanishes on the initial sphere $S_{1-\delta, \delta}$. This gives the desired estimates on $\|\underline{L}\phi\|_{L^\infty(S_{\underline{u}, \delta})}$.

For $(l, k) = (0, k)$, we can use (2.64) to obtain the desired estimates in a similar way.

We assume that for all $(l', k') < (l, k)$, we have

$$\begin{aligned} \|\nabla Z_b^{l'} Z_g^{k'-l'} \phi\|_{L^\infty(S_{\underline{u}, \delta})} &\lesssim \delta^{\frac{1}{4}} |\underline{u}|^{-\frac{3}{2} - \frac{\alpha}{4}} C(I_n), \\ \|L Z_b^{l'} Z_g^{k'-l'} \phi\|_{L^\infty(S_{\underline{u}, \delta})} &\lesssim \delta^{\frac{1}{4}} |\underline{u}|^{-2} C(I_n), \\ \|\underline{L} Z_b^{l'} Z_g^{k'-l'} \phi\|_{L^\infty(S_{\underline{u}, \delta})} &\lesssim \delta^{\frac{1}{4}} |\underline{u}|^{-1} C(I_n). \end{aligned}$$

For (l, k) , we now reduce the estimates to the above induction hypothesis. Because we have already proved the case for $(l, k) = (0, k)$, so we can assume in addition that $l \geq 1$.

We first bound $\nabla Z_b^l Z_g^{k-l} \phi$. In fact, we have

$$\begin{aligned} |\nabla Z_b^l Z_g^{k-l} \phi| &\lesssim \frac{1}{|\underline{u}|} |\Omega Z_b^l Z_g^{k-l} \phi| \\ &\leq \frac{1}{|\underline{u}|} \left(|Z_b \Omega Z_b^{l-1} Z_g^{k-l} \phi| + |[\Omega, Z_b] Z_b^{l-1} Z_g^{k-l} \phi| \right). \end{aligned}$$

Since $\Omega \in \mathcal{Z}_g$, we can use induction hypothesis (since we can reduce l), the first term is bounded by

$$\sum_{\partial \in \{L, \underline{L}, \nabla\}} \frac{1}{|\underline{u}|} |\partial Z_b^{l-1} Z_g^{k-l+1} \phi| \lesssim \delta^{\frac{1}{4}} |\underline{u}|^{-2} C(I_{n+1}).$$

For the second term, notice that $[\Omega, Z_b] = Z_b$, therefore, we have decreased the number k by 1. According to the induction hypothesis, it is bounded by

$$\sum_{\partial \in \{L, \underline{L}, \nabla\}} \frac{1}{|\underline{u}|} |\partial Z_b^{l-1} Z_g^{k-l} \phi| \lesssim \delta^{\frac{1}{4}} |\underline{u}|^{-2} C(I_{n+1}).$$

This gives the desired estimates for $\nabla Z_b^l Z_g^{k-l} \phi$.

We turn to the bound on $LZ_b^l Z_g^{k-l} \phi$. Evidently, it is bounded by $\sum_{\partial \in \{L, \underline{L}, \nabla\}} |L \partial Z_b^{l-1} Z_g^{k-l} \phi|$. When $\partial = \nabla$ in the sum, it can be bounded directly by the bound on $\nabla Z_b^{l-1} Z_g^{k-l+1} \phi$ derived above. Therefore, it suffices to bound $LLZ_b^{l-1} Z_g^{k-l} \phi$ and $L\underline{L}Z_b^{l-1} Z_g^{k-l} \phi$.

For $L\underline{L}Z_b^{l-1} Z_g^{k-l} \phi$, according to (2.20) (where we use $Z_b^{l-1} Z_g^{k-l}$ as commutator vector field), we have

$$-L\underline{L}Z_b^{l-1} Z_g^{k-l} \phi + \underline{L}Z_b^{l-1} Z_g^{k-l} \phi = \frac{1}{r} (LZ_b^{l-1} Z_g^{k-l} \phi - \underline{L}Z_b^{l-1} Z_g^{k-l} \phi) + \sum_{p+q \leq k-1} Q(\nabla Z^p \phi, \nabla Z^q \phi).$$

The second term on the left-hand side can be bounded by $\nabla Z_b^l Z_g^{k-l} \phi$. The terms on the right-hand side are all of lower degrees ($< k$) so that they are bounded by the induction hypothesis. Therefore, we have

$$\|L\underline{L}Z_b^{l-1} Z_g^{k-l} \phi\|_{L^\infty(S_{\underline{u}, \delta})} \lesssim \delta^{\frac{1}{4}} |\underline{u}|^{-2} C(I_{n+1}),$$

For $LLZ_b^{l-1} Z_g^{k-l} \phi$, we use the following identity:

$$LLZ_b^{l-1} Z_g^{k-l} \phi = L \left(\frac{1}{\underline{u}} (SZ_b^{l-1} Z_g^{k-l} \phi - \underline{u} \underline{L}Z_b^{l-1} Z_g^{k-l} \phi) \right).$$

The second term on the right-hand side can be bounded directly by the bound on $L\underline{L}Z_b^{l-1} Z_g^{k-l} \phi$ just derived. Therefore, it suffices to control the contribution from the first term, i.e.

$$\begin{aligned} L \left(\frac{1}{\underline{u}} (SZ_b^{l-1} Z_g^{k-l} \phi) \right) &= -\frac{1}{|\underline{u}|^2} SZ_b^{l-1} Z_g^{k-l} \phi + \frac{1}{\underline{u}} LSZ_b^{l-1} Z_g^{k-l} \phi \\ &= \underbrace{\frac{1}{|\underline{u}|^2} SZ_b^{l-1} Z_g^{k-l} \phi}_A + \underbrace{\frac{1}{\underline{u}} LZ_b^{l-1} Z_g^{k-l+1} \phi}_B. \end{aligned}$$

For A , by rewriting S as $\underline{u}L + u\underline{L}$, we can use induction hypothesis for $(l, k-1)$; for B , we can use induction hypothesis for $(l-1, k)$.

Hence, we have obtained the desired estimates for $LZ_b^l Z_g^{k-l} \phi$.

Finally, to bound $\underline{L}Z_b^l Z_g^{k-l} \phi$, we use the equation

$$-L\underline{L}Z_b^l Z_g^{k-l} \phi + \underline{L}Z_b^l Z_g^{k-l} \phi = -\frac{1}{r} (LZ_b^l Z_g^{k-l} \phi - \underline{L}Z_b^l Z_g^{k-l} \phi) + \sum_{p+q \leq k} Q(\nabla Z^p \phi, \nabla Z^q \phi).$$

We rewrite this as

$$\begin{aligned} &-L\underline{L}Z_b^l Z_g^{k-l} \phi - \frac{1}{r} \underline{L}Z_b^l Z_g^{k-l} \phi + Q(\nabla \phi, \nabla Z_b^l Z_g^{k-l} \phi) \\ &= -\underline{L}Z_b^l Z_g^{k-l} \phi - \frac{1}{r} LZ_b^{l-1} Z_g^{k-l} \phi + \sum_{\substack{p+q \leq k \\ p < k, q < k}} Q(\nabla Z^p \phi, \nabla Z^q \phi). \end{aligned}$$

All the terms on the right-side have been controlled in previous steps. Therefore, it is straightforward to see that the right-hand side is bounded by $C(I_{n+1})|\underline{u}|^{-2}\delta^{\frac{1}{4}}$. We now can mimic the proof for (2.17) by defining $y = \underline{u}\underline{L}Z_b^l Z_g^{k-l}\phi$, this leads to

$$|\underline{L}Z_b^l Z_g^{k-l}\phi(\underline{u}, \delta, \theta) - \frac{C(1-\delta)}{\underline{u}}\underline{L}Z_b^l Z_g^{k-l}\phi(1-\delta, \delta, \theta)| \lesssim \delta^{\frac{1}{4}}|\underline{u}|^{-1}C(I_{n+1}).$$

Taking into account of the vanishing property of $\underline{L}Z_b^l Z_g^{k-l}\phi$ on $S_{1-\delta, \delta}$, we complete the proof of Proposition 2.71.

Remark 2.8. For applications in the next section, we only need a slightly weakened (in decay) version of the estimates from Proposition 2.71:

$$\begin{aligned} \|\nabla Z_b^l Z_g^{k-l}\phi\|_{L^\infty(S_{\underline{u}, \delta})} + \|\underline{L}Z_b^l Z_g^{k-l}\phi\|_{L^\infty(S_{\underline{u}, \delta})} &\lesssim \delta^{\frac{1}{4}}|\underline{u}|^{-\frac{3}{2}}C(I_{n+1}), \\ \|\underline{L}Z_b^l Z_g^{k-l}\phi\|_{L^\infty(S_{\underline{u}, \delta})} &\lesssim \delta^{\frac{1}{4}}|\underline{u}|^{-1}C(I_{n+1}), \end{aligned} \quad (2.72)$$

where $0 \leq l \leq k \leq n-3$.

3. SMALL DATA REGION

In this section, we construct solutions in the entire small data region, i.e. region I. The approach is a modification of the classical approach with additional difficulties arising from the boundary C_δ .

3.1. Klainerman-Sobolev inequality revisited. We first introduce notations needed for the statement of the Klainerman-Sobolev inequality. We use Σ_t to denote the constant time slices in the small data region, i.e. for a fixed $t \in (1, +\infty)$,

$$\Sigma_t := \{(x, t) | t - r \geq \delta\}.$$

This is a ball of radius $t - \delta$. We recall that we use Σ_1 to denote the entire $t = 1$ hyperplane. Given a point $(t, x) \in \Sigma_t$ (assuming that $x \neq 0$), we use the $(t, B(t, x))$ to denote its corresponding boundary point, i.e. $(t, B(t, x))$ is the unique point on the boundary of Σ_t (also on C_δ) which is the intersection of the boundary of Σ_t with the ray emanated from $(t, 0)$ and passing from (t, x) . We now state the Klainerman-Sobolev inequality:

Proposition 3.1. For all $f \in C^\infty(\mathbb{R}^{3+1})$, $t > 1$ and a point (t, x) in the small data region, we have

$$|f(t, x)| \lesssim \frac{1}{(1+|u|)^{1/2}}|f(t, B(t, x))| + \frac{1}{(1+|\underline{u}|)(1+|u|)^{1/2}} \sum_{Z \in \mathcal{Z}, k \leq 3} \|Z^k f\|_{L^2(\Sigma_t)}. \quad (3.1)$$

We recall the following identities on \mathbb{R}^{3+1} :

$$\begin{aligned} \partial_t &= \frac{1}{t-r} \left(\frac{t}{t+r} S - \sum_{i=1}^3 \frac{x^i}{t+r} \Omega_{0i} \right), \\ \partial_i &= -\frac{1}{t-r} \left(\frac{x^i}{t+r} S - \frac{t}{t+r} \Omega_{0i} - \sum_{j=1}^3 \frac{x^j}{t+r} \Omega_{ij} \right), \\ \partial_r &= \frac{1}{t-r} \left(-\frac{r}{t+r} S + \sum_{i=1}^3 \frac{tx^i}{(t+r)r} \Omega_{0i} \right). \end{aligned} \quad (3.2)$$

Therefore, schematically, in terms of $Z_g \in \mathcal{Z}_g$, we write the above identities as

$$\partial = \frac{1}{|t-r|} Z_g$$

Near light cone C_0 . i.e. the hypersurface $t = r$, we can take Z to be ∂_i or ∂_t , therefore, schematically we have

$$\partial = \left(1 + \frac{1}{u}\right) Z.$$

We remark that this schematic expression means, for any function f , we have the following pointwise estimates:

$$|\partial f| \lesssim \left(1 + \frac{1}{u}\right) |Zf|.$$

We start the proof of (3.1). Let χ be a non-negative smooth cut-off function on $\mathbb{R}_{\geq 0}$ so that χ is supported in $[0, \frac{1}{2}]$ and $\chi \equiv 1$ on $[0, \frac{1}{4}]$. We decompose $f(t, x)$ as

$$\begin{aligned} f(t, x) &= f_1(t, x) + f_2(t, x) \\ &= \chi\left(\frac{x}{t}\right)f(t, x) + (1 - \chi\left(\frac{x}{t}\right))f(t, x). \end{aligned}$$

Therefore, the function $f_1(t, x)$ is supported in region

$$D_1 = \{(t, x) \mid 2r \leq t, t \geq 1\},$$

which is far away from the cone C_δ ; the function $\phi_2(t, x)$ is supported in region

$$D_2 = \{(t, x) \mid t - r \geq \delta, 4r \geq t, t \geq 1\}$$

which is close to the cone C_δ .

We first bound $f_1(t, x)$ in region D_1 . In the rest of the subsection, we regard t as a fixed large parameter. Let

$$\tilde{f}_1(x) = f_1(t, tx) = f(t, tx)\chi(x),$$

therefore, for a given positive integer m , we have

$$\begin{aligned} \|\partial^m \tilde{f}_1\|_{L^2(\mathbb{R}^3)}^2 &= \int_{\mathbb{R}^3} |\partial^m (f(t, tx)\chi(x))|^2 dx \\ &\lesssim \sum_{j=0}^m \int_{\mathbb{R}^3} |\partial^j (f(t, tx))|^2 |\nabla^{m-j} \chi|^2 dx \\ &\lesssim \sum_{j=0}^m \int_{\mathbb{R}^3} |t^j (\partial^j f)(t, tx)|^2 dx. \end{aligned}$$

Recall that we have $\partial = \frac{1}{1+|t-r|} Z$. In the region D_1 , we have $t \geq 2r$, hence $|t-r| \sim t$. Therefore, in D_1 , we have

$$|t\partial f| \lesssim |Zf|.$$

Thus, we have

$$\begin{aligned} \|\partial^m \tilde{f}_1\|_{L^2(\mathbb{R}^3)}^2 &\lesssim \sum_{j \leq m, Z \in \mathcal{Z}} \int_{\mathbb{R}^3} |Z^j f|^2(t, tx) dx \\ &= t^{-3} \sum_{j \leq m, Z \in \mathcal{Z}} \int_{\mathbb{R}^3} |Z^j f(t, y)|^2 dy. \end{aligned}$$

Therefore, according to the classical Sobolev inequality on \mathbb{R}^3 , we obtain

$$\|f_1\|_{L^\infty(\Sigma_t)} = \|\tilde{f}_1\|_{L^\infty(\Sigma_t)} \lesssim \frac{1}{t^{\frac{3}{2}}} \sum_{k \leq 2, Z \in \mathcal{Z}} \|Z^k f(t, \cdot)\|_{L^2(\Sigma_t)}. \quad (3.3)$$

We turn to the estimates on $f(t, x)$ in the region D_2 . On the hyperplane Σ_t , we draw a line from the origin and the point (x, t) . When a point moves along the radial direction on this line, it hits the characteristic boundary of C_δ at one point $(t, B(t, x))$. By integrating $\partial_r((1 + |t - r|)f^2(t, x))$ from $(t, B(t, x))$ to (t, x) , we obtain

$$\begin{aligned} (1 + |t - r|)f^2(t, x) &= (1 + \delta)f^2(t, B(t, x)) + \int_r^{t-\delta} \partial_r((1 + |t - r|)f^2(t, x))dr \\ &= (1 + \delta)f^2(t, B(t, x)) + \int_r^{t-\delta} -f^2(t, x) + 2(1 + |t - r|)f(t, x)\partial_r f(t, x)dr \end{aligned}$$

For the integrand in the last line, we apply the classical Sobolev inequalities on spheres $S_{t,r}$ (the sphere of radius r on Σ_t). Therefore, we obtain

$$\begin{aligned} ((1 + |t - r|)f^2(t, x) &\lesssim f^2(t, B(t, x)) + \int_r^{t-\delta} \frac{1}{r^2} \sum_{|\alpha| \leq 2} \|\Omega^\alpha f\|_{L^2(S_{t,r})}^2 dr \\ &\quad + \int_r^{t-\delta} \frac{(1 + |t - r|)}{r^2} \sum_{|\alpha|, |\beta| \leq 2} \|\Omega^\alpha f\|_{L^2(S_{t,r})} \|\Omega^\beta \partial_r f\|_{L^2(S_{t,r})} dr. \end{aligned}$$

Since $|t - r|\partial_r \lesssim Z$, we have

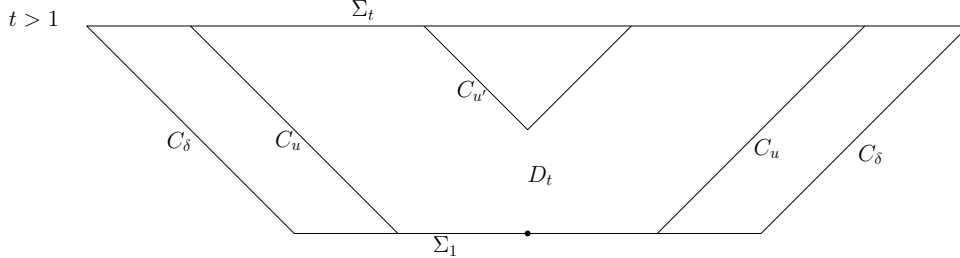
$$\begin{aligned} ((1 + |t - r|)f^2(t, x) &\lesssim f^2(t, B(t, x)) + \int_r^{t-\delta} \frac{1}{r^2} \sum_{k \leq 3, Z \in \mathcal{Z}} \|Z^k f\|_{L^2(S_{t,r})}^2 dr \\ &= f^2(t, B(t, x)) + \frac{1}{r^2} \sum_{k \leq 3, Z \in \mathcal{Z}} \|Z^k f(t, \cdot)\|_{L^2(\Sigma_t)}^2. \end{aligned} \tag{3.4}$$

The estimates (3.3) and (3.4) together give the desired estimates (3.1) and we complete the proof.

3.2. A priori energy estimates. For a $t \in (1, +\infty)$, we still use $\Sigma_t = \{(x, t) | t - r \geq \delta\}$ to denote the constant time slices in the small data region. For $k \in \mathbb{Z}_{\geq 0}$ and $t > 1$, we introduce the following energy norms:

$$\begin{aligned} \tilde{E}_k(t) &= \left(\sum_{Z \in \mathcal{Z}} \int_{\Sigma_t} |\partial_t Z^k \phi|^2 + \sum_{j=1}^3 |\partial_j Z^k \phi|^2 dx \right)^{\frac{1}{2}}, \\ \tilde{E}_{\leq k}(t) &= \left(\sum_{0 \leq j \leq k} \tilde{E}_j(t)^2 \right)^{\frac{1}{2}}. \end{aligned} \tag{3.5}$$

We use $D_{t,\delta}$ to denote the space-time region bounded by Σ_t , Σ_0 and C_δ . This region is obviously foliated by the constant time foliation $\{\Sigma_\tau | \tau \in [1, t]\}$ and this foliation is one of the foliations we use to derive energy estimates. The second foliation is the null foliation of outgoing null cones $\{C_u | u \in [\delta, t/2]\}$. This foliation is depicted as follows:



Whenever there is no confusion, we still use C_u to denote $C_u \cap D_t$. We use $D_{t,u}$ to denote the space-time region bounded by Σ_t , Σ_1 and C_u . This is a truncated solid light cone in \mathbb{R}^{3+1} . We use $\Sigma_{1,u}$ and $\Sigma_{t,u}$ to denote its bottom and top respectively. We remark that the bottom can be a single point.

Recall that (assuming that the solution ϕ exists up to time t), for $k \geq 0$ and $|\alpha| = k$, we have

$$\square Z^k \phi = \sum_{p+q \leq k} Q(\nabla Z^p \phi, \nabla Z^q \phi).$$

We multiply both sides by $\partial_t Z^k \phi$ and we then integrate over $D_{t,u}$. This leads to the following energy identity:

$$\begin{aligned} \int_{\Sigma_{t,u}} |\partial_t Z^k \phi|^2 + \sum_{j=1}^3 |\partial_j Z^k \phi|^2 &= \int_{\Sigma_{1,u}} |\partial_t Z^k \phi|^2 + \sum_{j=1}^3 |\partial_j Z^k \phi|^2 + \int_{C_u} |LZ^k \phi|^2 + |\nabla Z^k \phi|^2 \\ &\quad + \sum_{p+q \leq k} \iint_{D_{t,u}} Q(\nabla Z^p \phi, \nabla Z^q \phi) \partial_t Z^k \phi. \end{aligned}$$

Recall that we use $\partial \in \{L, \underline{L}, \nabla\}$ to denote a generic derivative and use $\partial_g \in \{L, \nabla\}$ to denote a good derivative. Therefore, by using $|\partial Z^k \phi|^2$ as a shorthand notation for $|\partial_t Z^k \phi|^2 + \sum_{j=1}^3 |\partial_j Z^k \phi|^2$ and using $|\partial_g Z^k \phi|^2$ as a shorthand notation for $|LZ^k \phi|^2 + |\nabla Z^k \phi|^2$, we have

$$\int_{\Sigma_{t,u}} |\partial Z^k \phi|^2 = \int_{\Sigma_{1,u}} |\partial Z^k \phi|^2 + \int_{C_u} |\partial_g Z^k \phi|^2 + \sum_{p+q \leq k} \iint_{D_{t,u}} Q(\nabla Z^p \phi, \nabla Z^q \phi) \partial_t Z^k \phi.$$

In applications, since the data prescribed on $\Sigma_{1,u}$ are trivial, we have

$$\int_{\Sigma_{t,u}} |\partial Z^k \phi|^2 = \int_{C_u} |\partial_g Z^k \phi|^2 + \sum_{p+q \leq k} \iint_{D_{t,u}} Q(\nabla Z^p \phi, \nabla Z^q \phi) \partial_t Z^k \phi. \quad (3.6)$$

Before we state the main estimates of the section, we first compute the energy flux $\int_{C_\delta} |\partial_g Z^k \phi|^2$ through the outermost cone C_δ . According to (2.71), for $k \leq n-2$, we have $|\partial_g Z^k \phi| \lesssim \delta^{\frac{1}{4}} |\underline{u}|^{-\frac{3}{2}-\frac{\alpha}{4}} C(I_{n+1})$, therefore,

$$\int_{C_\delta} |\partial_g Z^k \phi|^2 \lesssim \delta^{\frac{1}{2}} C(I_{n+1}), \quad (3.7)$$

where we still use $C(I_{n+1})$ to denote $C(I_{n+1})^2$.

Proposition 3.2. *Under the same assumptions as in the previous section, for sufficiently small δ , there exists a unique global future in time solution ϕ of (1.2) on the small data region, so that together with the solution constructed in the short pulse region, we have a unique future in time solution ϕ . Moreover, this solution ϕ on the small data region enjoys the following energy estimates:*

$$\tilde{E}_{\leq n}(t) \lesssim \delta^{\frac{1}{4}} C(I_{n+1}), \quad (3.8)$$

for all $t > 1$,

Remark 3.3. *The existence of solutions in the small data region follows from the a priori estimate (3.8). Since the argument is routine, we will not pursue this point here.*

We use a bootstrap argument to prove the proposition. We assume that the solution exists up to time t and for all $1 \leq t' \leq t$, we have

$$\tilde{E}_{\leq 7}(t') \lesssim M \delta^{\frac{1}{4}}. \quad (3.9)$$

It suffices to show that we can indeed choose M so that it depends only on I_n

We first point out that we can derive L^∞ bound on $\partial Z^p \phi$ for $p \leq 4$. According to Klainerman-Sobolev inequality, we have

$$\begin{aligned} |\partial Z^p \phi(\tau, x)| &\lesssim \frac{1}{(1+|u|)^{1/2}} |\partial Z^p \phi(\tau, B(\tau, x))| + \frac{1}{(1+|\underline{u}|)(1+|u|)^{1/2}} \sum_{Z \in \mathcal{Z}, l \leq 3} \|Z^l \partial Z \phi\|_{L^2(\Sigma_\tau)} \\ &\lesssim \frac{C(I_{n+1})}{(1+\underline{u})(1+u)^{1/2}} \delta^{\frac{1}{4}} + \frac{M}{(1+\underline{u})(1+|u|)^{1/2}} \delta^{\frac{1}{4}}. \end{aligned}$$

In particular, based on (3.2), it is well known that for good derivatives ∂_g , we have

$$|\partial_g Z^p \phi(\tau, x)| \lesssim \frac{M}{t^{\frac{3}{2}}} \delta^{\frac{1}{4}}.$$

For all $u \geq \delta$, according to (3.6), we have

$$\begin{aligned} \int_{C_u} |\partial_g Z^k \phi|^2 &\leq \int_{\Sigma_{t,u}} |\partial Z^k \phi|^2 + \sum_{p+q \leq k} \iint_{D_{t,u}} |Q(\nabla Z^p \phi, \nabla Z^q \phi) \partial_t Z^k \phi| \\ &\leq \int_{\Sigma_t} |\partial Z^k \phi|^2 + \sum_{p+q \leq k} \iint_{D_{t,\delta}} |Q(\nabla Z^p \phi, \nabla Z^q \phi) \partial_t Z^k \phi|. \end{aligned}$$

For the last step, we have enlarged the domain for integration. Therefore, according to the foliation $\{u \in [\delta, \frac{t}{2}] \mid C_u\}$, for the given constant $\varepsilon_0 \in (0, \frac{1}{2})$, we have

$$\begin{aligned} \iint_{D_{t,\delta}} \frac{|\partial_g Z^k \phi|^2}{(1+|u|)^{1+\varepsilon_0}} &= \int_\delta^{t/2} \frac{1}{(1+|u|)^{1+\varepsilon_0}} \left(\int_{C_u} |\partial_g Z^k \phi|^2 \right) du \\ &\leq \int_\delta^{t/2} \frac{1}{(1+|u|)^{1+\varepsilon_0}} \left(\int_{\Sigma_t} |\partial Z^k \phi|^2 + \sum_{p+q \leq k} \iint_{D_{t,\delta}} |Q(\nabla Z^p \phi, \nabla Z^q \phi) \partial_t Z^k \phi| \right) du' \end{aligned}$$

Since the quantity inside the parenthesis is independent of u' , we obtain

$$\iint_{D_{t,\delta}} \frac{|\partial_g Z^k \phi|^2}{(1+|u|)^{1+\varepsilon_0}} \lesssim \int_{\Sigma_t} |\partial Z^k \phi|^2 + \sum_{p+q \leq k} \iint_{D_{t,\delta}} |Q(\nabla Z^p \phi, \nabla Z^q \phi) \partial_t Z^k \phi|. \quad (3.10)$$

We take $u = \delta$ in (3.6). In view of (3.7), we obtain immediately that

$$\begin{aligned} \int_{\Sigma_t} |\partial Z^k \phi|^2 &\leq \int_{C_\delta} |\partial_g Z^k \phi|^2 + \sum_{p+q \leq k} \iint_{D_{t,\delta}} |Q(\nabla Z^p \phi, \nabla Z^q \phi)| |\partial_t Z^k \phi| \\ &\lesssim \delta^{\frac{1}{2}} C(I_{n+1}) + \sum_{p+q \leq k} \iint_{D_{t,\delta}} |Q(\nabla Z^p \phi, \nabla Z^q \phi)| |\partial_t Z^k \phi|. \end{aligned} \quad (3.11)$$

Together with (3.10), we have

$$\iint_{D_{t,\delta}} \frac{|\partial_g Z^k \phi|^2}{(1+|u|)^{1+\varepsilon_0}} \lesssim \delta^{\frac{1}{2}} C(I_{n+1}) + \sum_{p+q \leq k} \iint_{D_{t,\delta}} |Q(\nabla Z^p \phi, \nabla Z^q \phi)| |\partial_t Z^k \phi|. \quad (3.12)$$

In view of (3.11), we arrive at the following energy estimates:

$$\int_{\Sigma_t} |\partial Z^k \phi|^2 + \iint_{D_{t,\delta}} \frac{|\partial_g Z^k \phi|^2}{(1+|u|)^{1+\varepsilon_0}} \lesssim \delta^{\frac{1}{2}} C(I_{n+1}) + \sum_{p+q \leq k} \iint_{D_{t,\delta}} |Q(\nabla Z^p \phi, \nabla Z^q \phi)| |\partial_t Z^k \phi|.$$

By summing over k , we finally obtain that

$$\tilde{E}_{\leq k}(t) + \sum_{\substack{l \leq k, \\ Z \in \mathcal{Z}}} \iint_{D_{t,\delta}} \frac{|\partial_g Z^l \phi|^2}{(1+|u|)^{1+\varepsilon_0}} \lesssim \delta^{\frac{1}{2}} C(I_{n+1}) + \sum_{\substack{l \leq k, p+q \leq l, \\ Z \in \mathcal{Z}}} \iint_{D_{t,\delta}} |Q(\nabla Z^p \phi, \nabla Z^q \phi)| |\partial_t Z^l \phi|. \quad (3.13)$$

Since we have the energy term $\tilde{E}_{\leq k}(t)$ on the left-hand side, to complete the bootstrap argument, it suffices to control the second term on the right-hand side. According to the structure of null forms, this term is bounded by

$$\sum_{\substack{l \leq k, p+q \leq l, \\ Z \in \mathcal{Z}}} \iint_{D_{t,\delta}} |\partial_g Z^p \phi| |\partial Z^q \phi| |\partial_t Z^l \phi|.$$

According to whether $p < q$ or $p \geq q$, we break this term into two pieces (we replace ∂_t by ∂):

$$S_1 + S_2 = \sum_{\substack{l \leq k, p+q \leq l, \\ p < q, Z \in \mathcal{Z}}} \iint_{D_{t,\delta}} |\partial_g Z^p \phi| |\partial Z^q \phi| |\partial Z^l \phi| + \sum_{\substack{l \leq k, p+q \leq l, \\ p \geq q, Z \in \mathcal{Z}}} \iint_{D_{t,\delta}} |\partial_g Z^p \phi| |\partial Z^q \phi| |\partial Z^l \phi|.$$

For S_1 , since $p < q$, we have $k - p \geq \lfloor \frac{1}{2}k \rfloor \geq 3$. Here $\lfloor \frac{1}{2}k \rfloor$ denotes the largest integer less or equal to $\frac{1}{2}$. We can apply the L^∞ estimates for good derivatives $\partial_g Z^p \phi$. Therefore,

$$\begin{aligned} S_1 &\lesssim \sum_{\substack{l \leq k, p+q \leq l, \\ p < q, Z \in \mathcal{Z}}} \int_1^t \frac{M}{\tau^{\frac{3}{2}}} \delta^{\frac{1}{4}} \|\partial Z^q \phi\|_{L^2(\Sigma_\tau)} \|\partial Z^l \phi\|_{L^2(\Sigma_\tau)} d\tau \\ &\lesssim M^3 \delta^{\frac{3}{4}}. \end{aligned}$$

For S_2 , we apply Klainerman-Sobolev to $|\partial Z^q \phi|$ and we obtain

$$\begin{aligned} S_2 &\lesssim \sum_{\substack{l \leq k, p+q \leq l, \\ p \geq q, Z \in \mathcal{Z}}} \iint_{D_{t,\delta}} \frac{M}{t(1+|u|)^{\frac{1}{2}}} \delta^{\frac{1}{4}} |\partial_g Z^p \phi| |\partial Z^l \phi| \\ &\lesssim \epsilon \sum_{p \geq k, Z \in \mathcal{Z}} \iint_{D_{t,\delta}} \frac{|\partial_g Z^p \phi|^2}{(1+|u|)^{1+\varepsilon_0}} + \frac{1}{\epsilon} \sum_{\substack{l \leq k, p+q \leq l, \\ p \geq q, Z \in \mathcal{Z}}} \iint_{D_{t,\delta}} \frac{M^2(1+|u|)}{t^2} \delta^{\frac{1}{2}} |\partial Z^l \phi|^2 \\ &\lesssim \epsilon \sum_{p \geq k, Z \in \mathcal{Z}} \iint_{D_{t,\delta}} \frac{|\partial_g Z^p \phi|^2}{(1+|u|)^{1+\varepsilon_0}} + \frac{1}{\epsilon} M^4 \delta, \end{aligned}$$

where the constant ϵ will be determined later on.

Back to (3.13), the estimates on S_1 and S_2 yield

$$\tilde{E}_{\leq k}(t) + \sum_{\substack{l \leq k, \\ Z \in \mathcal{Z}}} \iint_{D_{t,\delta}} \frac{|\partial_g Z^l \phi|^2}{(1+|u|)^{1+\varepsilon_0}} \lesssim \delta^{\frac{1}{2}} C(I_{n+1}) + M^3 \delta^{\frac{3}{4}} + \epsilon \sum_{p \geq 7, Z \in \mathcal{Z}} \iint_{D_{t,\delta}} \frac{|\partial_g Z^p \phi|^2}{(1+|u|)^{1+\varepsilon_0}} + \frac{1}{\epsilon} M^4 \delta.$$

By choosing a suitable small constant ϵ , we can remove the integral term on the right-hand side and obtain

$$\tilde{E}_{\leq k}(t) + \sum_{\substack{l \leq k, \\ Z \in \mathcal{Z}}} \iint_{D_{t,\delta}} \frac{|\partial_g Z^l \phi|^2}{(1+|u|)^{1+\varepsilon_0}} \lesssim \delta^{\frac{1}{2}} C(I_{n+1}) + M^3 \delta^{\frac{3}{4}} + \frac{1}{\epsilon} M^4 \delta.$$

Hence,

$$\tilde{E}_{\leq k}(t) \lesssim \delta^{\frac{1}{2}} C(I_{n+1}) + M^3 \delta^{\frac{3}{4}} + \frac{1}{\epsilon} M^4 \delta.$$

We then can choose a sufficiently small δ and this completes the bootstrap argument.

ACKNOWLEDGMENT

S.Miao is supported by NSF grant DMS-1253149 to The University of Michigan; L.Pei is supported by the PhD research fellowship of Norwegian University of Science and Technology; P.Yu is supported by NSFC 11101235 and NSFC 11271219. S.Miao would like to thank Sohrab Shahshahani for helpful comments on a previous version of this manuscript. P.Yu would like to thank Sergiu Klainerman for the communication on the relaxation of the propagation estimates.

REFERENCES

- [1] D. Christodoulou, *Global solutions of nonlinear hyperbolic equations for small initial data*, Comm. Pure Appl. Math., 39(2), 267-282, 1986.
- [2] D. Christodoulou, *The Formation of Black Holes in General Relativity*, Monographs in Mathematics, European Mathematical Soc. 2009.
- [3] D. Christodoulou and S. Klainerman, *The Global Nonlinear Stability of Minkowski space*, Princeton Mathematical Series 41, 1993.
- [4] F. John, *Blow-up of solutions of nonlinear wave equations in three space dimensions*. Manuscripta Math., 28(1-3), 235-268, 1979.
- [5] S. Klainerman, *Global existence for nonlinear wave equations*, Comm. Pure Appl. Math., 33(1):43-101, 1980.
- [6] S. Klainerman, *Long time behaviour of solutions to nonlinear wave equations*, Proceedings of the International Congress of Mathematicians, 1209 - 1215, Warsaw, 1984. PWN.
- [7] S. Klainerman. *Uniform decay estimates and the Lorentz invariance of the classical wave equation*, Comm. Pure Appl. Math., 38(3):321-332, 1985.
- [8] S. Klainerman and I. Rodnianski, *On the Formation of Trapped Surfaces*, Acta Math. 208 (2012), no. 2, 211C333.
- [9] A. D. Rendall, *Reduction of the characteristic initial value problem to the Cauchy problem and its applications to the Einstein equations*, Proc. Roy. Soc. Lond. A 427, 221 - 239 (1990).
- [10] J. Wang and P. Yu *A Large Data Regime for non-linear Wave Equations*, arXiv:1210.2056,
- [11] J. Wang and P. Yu *Long time solutions for wave maps with large data*, J. Hyperbolic Differ. Equ. 10, no. 2, 371-414, 2013.
- [12] S. Yang *Global solutions of nonlinear wave equations with large energy*, arXiv:1312.7265,